Advances in Computational Mathematics and Modelling Website:https://anandice.ac.in/acmmjournal/index.html
Volume 1, Number 1, 2023, 1–7 ISSN:2583-9357

FROM ABEL CONTINUITY THEOREM TO PALEY-WIENER THEOREM

Yu Sun[1], Praveen Agarwal[2], Takako Kuzumaki[3,†] and Shigeru Kanemitsu[4]

Abstract In this note we reveal that the missing link among a few crucial results in analysis, Abel continuity theorem, convergence theorem on (generalized) Dirichlet series, Paley-Wiener theorem is the Laplace transform with Stieltjes integration. By this discovery, the reason why the domains of Stoltz path and of convergence look similar is made clear. Also as a natural intrinsic property of Stieltjes integral, the use of partial summation in existing proofs is elucidated. Secondly, we shall reveal that a basic part of the proof of Paley-Wiener theorem is a version of the Laplace transform.

Keywords Laplace transform; Stieltjes integral; Abel continuity theorem; Paley-Wiener theorem; conformal mapping

MSC(2010) 130E99; 44A10; 40A05

1. Introduction

Let $\{\lambda_n\} \subset \mathbb{R}$ be an increasing sequence for which we may suppose $\lambda_1 > 0$. For a sequence $\{a_n\} \subset \mathbb{C}$, the series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{1.1}$$

convergent in some half-plane, is called a generalized Dirichlet series.
1. If $\lambda_n = \log n$ with log denoting the principal value, $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is (an ordinary) Dirichlet series.
2. If $\lambda_n = n$ and $e^{-s} = w$, $f(w) = f(-\log w) = \sum_{n=1}^{\infty} a_n w^n$ is the power series.

In all literature [2], [8], [9], etc. the convergence theorem for generalized Dirichlet series, Theorem 1.1 and the Abel continuity theorem, Corollary 1.1 are regarded as independent and proofs are given separately. Cf. also [10] (cf. [4]). In [5] it is shown that Theorem 1.1 entails Corollary 1.1 via a counterpart, Corollary 1.2 together with conformality of the analytic mapping $e^{-s} = w$, thus revealing the reason why the convergence domains are angular domains of a similar shape. The

†the corresponding author. Email address:kuzumaki@gifu-u.ac.jp
1Taishan College, Shandong University, Jinan, 250110, P.R.China
2Anand International College of Engineering, Near Kanota, Agra Road, Jaipur-303012, Rajasthan, India
3Faculty of Engineering Gifu University Gifu 501-1193, Japan
4Graduate School of Engineering Kyushu Inst. Tech. Kitakyushu 804-8555, Japan

proof uses a general form of the partial summation [5, Lemma 2] for a generalized sequence $\{\lambda_n\}$, thus unifying all existing proofs.

In this note we employ a general treatment by (Lebesgue-) Stieltjes integrals to attain two objects at a stretch. I.e. we follow [13] to introduce Corollary 1.3 whose discrete version leads to Theorem 1.1. In the proof, integration by parts is used which is a more general version of the partial summation. Then on one hand we cover Abel continuity theorem by the convergence theorem, Corollary 1.3, for Laplace transforms and conformality, revealing the reason why convergence domains being similar.

On the other hand, we shall show that the basic part of the Paley-Wiener theorem (cf. e.g. [3]) is laid by the Laplace transform method. Then we appeal to two fundamental results, the Plancherel formula and the Fourier inversion formula to conclude the theorem. By finding this hidden link of Laplace transform, we are able to treat these two remote-looking objects of Paley-Wiener theorem and Abel continuity theorem in a unified way, up to some auxiliary fundamental results. The Paley-Wiener theorem has recently been highlighted in view of its essential application to signal restoration. In both well-known approaches by sampling [13], [6], [11] and by Bernstein polynomials [1] the Paley-Wiener theorem plays a fundamental role.

Theorem 1.1. *If the series* (1.1) *is convergent for* $s = s_0 = \sigma_0 + it_0$*, then* $f(s)$ *is uniformly convergent in the right half-plane* $\sigma > \sigma_0$ *in the wide sense and represents an analytic function there. More precisely, let* D *be an angular domain*

$$\sigma - \sigma_0 \geq 0, \quad \arg(s - s_0) \leq \delta \tag{1.2}$$

with $0 < \delta < \frac{\pi}{2}$*. Then* $f(s)$ *is uniformly convergent on* D *in the wide sense.*

Corollary 1.1. (Abel continuity theorem) *Suppose a power series* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *converges at the point* z_0 *on its circle of convergence. Draw two chords (inside the circle) that start from* z_0 *and form an angle* δ *with the tangent at* z_0 *of the circle* $(0 < \delta < \frac{\pi}{2})$*. Let* Δ *be the (closure of) intersection of this angular subdomain and the disc of convergence. Then* $f(z)$ *approaches* $f(s_0)$ *as* $z \to z_0$ *in the angular domain inside* Δ*. This is often said as* z *approaches to* z_0 **along Stoltz path***.*

Corollary 1.2. (Counterpart of Abel continuity theorem) $f(s)$ *approaches to* $f(s_0)$ *as* $s \to s_0$ *in the angular domain* (1.2)*.*

Lemma 1.1.

(i) *The Stieltjes integral* $\int_a^b f \, \mathrm{d}g$ *exists if* f *is continuous and* g *is of bounded variation and linear in* f *and* g*. The role can be changed in view of Item* (ii)*. It holds that*

$$\int_a^b \mathrm{d}g(x) = g(b) - g(a). \tag{1.3}$$

(ii) *The formula for integration by parts holds true:*

$$\int_a^b f(x) \, \mathrm{d}g(x) = [f(x)g(x)]_a^b - \int_a^b g(x) \, \mathrm{d}f(x), \tag{1.4}$$

provided that f is continuous and g is of bounded variation or g is continuous and f is of bounded variation.

(iii) *If g is a step function with jumps a_n at x_n, the Stieltjes integral reduces to the sum:*

$$\int_a^b f(x)\,\mathrm{d}g(x) = \sum_{a<x_n\leq b} f(x_n)a_n. \tag{1.5}$$

(iv) *If $f(x)$ is continuous, $\varphi(x) \in L = L^1$ on $[a,b]$ and*

$$g(x) = \int_c^x \varphi(u)\mathrm{d}u \quad x, c \in [a,b],$$

then

$$\int_a^b f(x)\mathrm{d}g(x) = \int_a^b f(x)\varphi(x)\mathrm{d}x = \int_a^b f(x)g'(x)\mathrm{d}x, \tag{1.6}$$

where the last integral exists as a Lebesgue integral.

Lemma 1.2. *Let*

$$f(s) = \int_0^\infty e^{-sx}\mathrm{d}g(x) \tag{1.7}$$

and

$$h(u) = \int_0^u e^{-s_0x}\mathrm{d}g(x) \quad u \geq 0. \tag{1.8}$$

If

$$\limsup_{0\leq u} |h(u)| = M < \infty \tag{1.9}$$

with $s_0 = \sigma_0 + it_0$, then (1.7) is convergent at s_0 and

$$\int_0^\infty e^{-sx}\mathrm{d}g(x) = (s-s_0)\int_0^\infty e^{-(s-s_0)x}\mathrm{d}h(x) \tag{1.10}$$

the integral on the right being absolutely convergent.

Proof. By Lemma 1.1, we have successively for $X < Y$

$$\begin{aligned}\int_X^Y e^{-sx}\mathrm{d}g(x) &= \int_X^Y e^{-(s-s_0)x}\mathrm{d}h(x) \\ &= \left[e^{-(s-s_0)x}h(x)\right]_X^Y + (s-s_0)\int_X^Y e^{-(s-s_0)x}h(x)\mathrm{d}x.\end{aligned} \tag{1.11}$$

For $X = 0$, $Y \to \infty$, the above tends to $\to (s-s_0)\int_0^\infty e^{-(s-s_0)x}h(x)\mathrm{d}x$ where the passage to the limit follows from

$$\left|(s-s_0)\int_0^\infty e^{-(s-s_0)x}h(x)\mathrm{d}\right| \leq M|s-s_0|\int_0^\infty e^{-(\sigma-\sigma_0)x}\mathrm{d}x \leq M\frac{|s-s_0|}{\sigma-\sigma_0} \tag{1.12}$$

for $\sigma > \sigma_0$. The convergence is absolute (and uniform). □

Remark 1.1. From Lemma 1.2 it follows that the domain of convergence is a half-plane. For our purpose we need a stronger results that follow. In most of existing literature on (generalized) Dirichlet series, the above proof is used with integration by parts (1.11) as partial summation.

Theorem 1.2. [12, Theorem 4.3, p. 54] *If the integral* (1.7) *is convergent at* $s = s_0$ *and* $H > 0, K > 1$ *are constants, then the integral is uniformly convergent in the domain*

$$D : |s - s_0| \le K(\sigma - \sigma_0)e^{H(\sigma-\sigma_0)}, \quad \sigma \ge \sigma_0. \tag{1.13}$$

In the proof, (1.11) with $Y = \infty$ is essentially used.

Corollary 1.3. *If the integral*

$$f(s) = \int_0^\infty e^{-sx} \mathrm{d}g(x) \tag{1.14}$$

is convergent at $s = s_0 = \sigma_0 + it_0$ *and* $K = \frac{1}{\cos\delta} > 1$ *where* δ *has the same meaning in Theorem 1.1, then the integral is uniformly convergent in the angular domain*

$$D_{s_0} : |s - s_0| \le K(\sigma - \sigma_0), \quad \sigma \ge \sigma_0. \tag{1.15}$$

Suppose $f(z)$ is an integral function satisfying the conditions

$$|f(z)| \le Ce^{A|z|}, \qquad 0 < A \tag{1.16}$$

and

$$f(x) = o(1) \quad \text{as} \quad z = x \to +\infty. \tag{1.17}$$

For $\alpha \in \mathbb{R}$ let $\Gamma_\alpha(r)$ denote a ray starting from the origin

$$\Gamma_\alpha = \Gamma_\alpha(r) : z = re^{i\alpha}, \quad 0 \le r < \infty. \tag{1.18}$$

Let

$$P_\alpha = \{s | \operatorname{Re}(se^{\alpha}) > A\}, \tag{1.19}$$

so that P_α is the **rotated right half-plane** obtained from the right half-plane $\{s | \operatorname{Re} s > A\}$ by rotation $-\alpha$ in the positive direction. Let $\Phi_\alpha(w)$ be defined by

$$\Phi_\alpha(s) = \Phi_\alpha(f, s) = \int_{\Gamma_\alpha} e^{-sz} f(z)\, \mathrm{d}z, \quad s \in P_\alpha. \tag{1.20}$$

In view of (1.18) and (1.19) the Weierstrass M-test applies and therefore the integral (1.20) converges absolutely and represents an analytic function in P_α for every $\alpha \in \mathbb{R}$. Φ_0 is the Laplace transform

$$\Phi_0(f, s) = \mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x)\, \mathrm{d}x, \quad \sigma := \operatorname{Re} s > a. \tag{1.21}$$

The integral (1.21) is convergent at $s = 0$ in view of (1.17). Hence Corollary 1.3 applies and is is analytic in the angular domain (1.15).

Lemma 1.3. *Suppose* $0 < \beta - \alpha < \pi$. *Then in the intersection* $P_\alpha \cap P_\beta$, *we have*

$$\Phi_\alpha(f, s) = \Phi_\beta(f, s). \tag{1.22}$$

Proof. We modify the proof in [7, p.376]. It suffices to prove this in the case α and 0 ($\alpha > 0$, say,) by rotating the configuration by $-\alpha$ and writing α for $\beta - \alpha$. If we prove this case, then we also prove that $\Phi_\alpha(e)$ is an integral expression of the Laplace transform. Let

$$s_{\alpha/2} = |s|e^{-\frac{\alpha}{2}i} \tag{1.23}$$

be a ray. Then

$$\mathrm{Re}(se^{\alpha}) = \mathrm{Re}(s) = |s|\cos\frac{\alpha}{2}.$$

Hence if

$$|s| > \frac{A}{\cos\frac{\alpha}{2}}, \tag{1.24}$$

then $s_{\alpha/2} \in P_\alpha \cap P_\beta$. Hence on the circular arc $C_R : z = Re^{i\theta}$, $0 \le \theta \le \alpha$, we have

$$\begin{aligned} |f(z)e^{-s_{\alpha/2}z}| &= O\left(\exp\left(AR\right)\exp(-r\cos(\theta-\alpha/2))\right) \\ &= O\left(\exp\left(A - |s|\cos(\alpha/2)\right)R\right) \to 0, \quad R \to \infty \end{aligned} \tag{1.25}$$

as long as (1.24). The Cauchy integral theorem applied to the sector with boundaries $[0, R]$, C_R and $z = re^{i\alpha}$, $r : R \to 0$ shows that the sum of integrals along the rays and the integral along the arc is 0. By (1.25), $\Phi_\alpha = \Phi_0$ on the ray (1.23) with (1.24). But then by the consistency theorem, they must coincide in $P_\alpha \cap P_0$. □

Theorem 1.3. *Every* $\Phi_\alpha(f, s)$ *in* (1.20) *is an integral representation of the Laplace transform* (1.21) *for all* $\alpha \in \mathbb{R}$.

Indeed, for $|\alpha| \le \frac{\pi}{2}$, theorem reduces to Corollary 1.3 and Lemma 1.3. And Φ_α coincides with Φ_0 in the angular domain D_{s_0} in (1.15) and in $P_\alpha \backslash D_{s_0}$, Φ_α gives an analytic continuation.

For $\frac{\pi}{2} < |\alpha| \le \pi$, every $\Phi_\alpha(f, s)$ is an analytic continuation of $\Phi_(f, s)$. This also follows from the Schwarz reflection principle with respect to the imaginary axis.

2. Paley-Wiener theorem

In this section we elucidate the proof of the Paley-Wiener theorem given in Rudin [7]. There are some books and papers related to this theorem (cf. e.g. [3, pp.38-40]).

Let

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} F(\omega)e^{i\omega z}\,\mathrm{d}\omega, \tag{2.1}$$

where $A > 0$ and $F \in L^2(-A, A)$. One can show that f is an integral function. It satisfies the growth condition

$$|f(z)| \le \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} |F(\omega)|e^{-\omega y}\,\mathrm{d}\omega \le \frac{1}{\sqrt{2\pi}} e^{A|y|} \int_{-A}^{A} |F(\omega)|\,\mathrm{d}\omega, \tag{2.2}$$

where we write $z = x + iy$. Hence denoting the last integral by C, we deduce (1.16).

Definition 2.1. An integral function f that satisfies condition (1.16) is said to be of exponential type (or of order 1 à la Hadamard).

Thus we have

Proposition 2.1. *Every f of the form (2.1) is an integral function which satisfies (1.16) and whose restriction is in L^2 (by the Plancherel theorem).*

The remarkable theorem of Paley-Wiener asserts that the converse also holds.

Theorem 2.1. (Paley-Wiener theorem) *Suppose A and C are positive constants, that f is an* **integral function of exponential type**, *i.e. satisfying* (1.16) *for all values of z, and the boundary condition*

$$\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x < \infty. \tag{2.3}$$

Then there exists a boundary function $F \in L^2(-\infty, \infty)$ such that

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} F(\omega) e^{i\omega z} \,\mathrm{d}\omega \tag{2.4}$$

for **all** *values of z.*

Proof. With enough preparation given toward the end of the previous section, the proof now goes along lines of proof in Rudin. Condition (1.17) is assured by (2.3). The key lies in the proof of

$$\lim_{\delta \to 0} \int_{-\infty}^{\infty} f_\delta(x) e^{-itx} \,\mathrm{d}x = 0, \quad t \in \mathbb{R}, \quad |t| > A, \tag{2.5}$$

where

$$f_\delta(t) = f(x) e^{-\delta|x|}. \tag{2.6}$$

Since $f_\delta \to f$ in L^2-norm, the Plancherel theorem implies that the Fourier transforms of f_δ tends to $\hat{f}$ in L^2-norm. (2.5) shows $\hat{f}$ vanishes in $|t| > A$ and the Fourier inversion formula entials (2.4) for a.a. values. But then both sides being analytic, hence for all values.

Proof of (2.5) follows from Lemma 1.3. For first note that

$$\int_{-\infty}^{\infty} f_\delta(x) e^{-itx} \,\mathrm{d}x = \Phi_0(f_\delta) - \Phi_\pi(f_{-\delta}). \tag{2.7}$$

Then rewrite this as

$$\int_{-\infty}^{\infty} f_\delta(x) e^{-itx} \,\mathrm{d}x = \begin{cases} \Phi_{-\frac{\pi}{2}}(f_\delta) - \Phi_{-\frac{\pi}{2}}(f_{-\delta}) & t > A \\ \Phi_{\frac{\pi}{2}}(f_\delta) - \Phi_{\frac{\pi}{2}}(f_{-\delta}) & t < -A \end{cases} \tag{2.8}$$

Then (2.5) immediately follows, completing the proof. □

Conclusion. By applying the Laplace transform with Stieltjes integratioo, we have established the following theorem.

Theorem 2.2. *Both the Abel continuity theorem (Corollary 1.1) and the Paley-Wiener theorem (Theorem 2.1) are based on similar grounds and may be treated in a unified way up to some auxiliary fundamental results, where in the former we use the discrete form as partial summation and in the latter, a generalized Laplace transform along a ray.*

Aknowledgement. We thank our affiliations for faclities and help. We also thank the referee for useful comments.
Conflict of interest. There is no conflict of interest with other works.

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