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NON LINEAR FRACTIONAL BOUNDARY VALUE PROBLEM UNDER ATANGANA-BALEANU DERIVATIVE

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Abstract In this work, we investigate the existence and uniqueness of solutions to nonlinear fractional boundary value problems under the Atangana-Baleanu derivative in the sense of Caputo (ABC for short), with the boundary conditions depending on the ABC derivative. To accomplish this, we will use the ABC definitions and their properties to the order of derivation $\delta \in [n, n+1]$, as well as the Banach fixed point theorem, to demonstrate our goal.

Keywords Boundary value problem, Fractional calculus, Atangana-Baleanu derivative, Fractional integral operators, Banach fixed point theorem.

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1. Introduction

Since the final years of the 17th century, the notion of fractional calculus, which is a generalization of conventional differentiation and integration to any order, including non-integer orders, has had a long history. In a letter to L'Hospital, Leibniz revealed his interest in this matter almost as soon as the concepts of classical calculus were understood in 1695 when Newton and Leibniz built the foundations of differential and integral calculus. He proposed to generalize the notation $d^{1/2}h$ which means the derivative of function h with order 1/2 and a formula for the n-th derivative of the product of two functions for $n > 0$ [1]. Then, many researchers devote themselves to this field, we mention Euler in 1730. Liouville in 1847, so that was the first one he proposed a definition of the fractional integral from a generalization of Taylor's formula [2]. Subsequently, several definitions of fractional integrals and derivatives appeared and were developed. We find among them Rieman-Liouville [3] , Caputo [4, 5], Hadamard [6, 7], Caputo-Fabrizio [8], and Atangana-Baleanu derivatives [9]. Despite their multiplicity, it cannot be said that one is better than the other. Rather, each of them has its characteristics, and each one complements the other. They also generalize it to include various aspects and all of them have proven to be distinctive tools for integrating and modeling many phenomena.

Applications of fractional differential equations using these multiple types of derivatives have increased in the past few decades after some small developments. It has produced better results in different field like engineering, biology, and epidemiology [10, 11]. For example, in [12] the authors analyze the impact of diabetes

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and resistant strains in a model for TB infection. Agarwal et al. also apply the Caputo fractional derivative in a variety of sciences, for more details, see [13–15]. Whereas in [16], other authors give the numerical solution of some fractional dynamical systems in medicine involving the Caputo and AB derivatives. Moreover, they showed that the integer order is less accurate than the fractional order.

In this paper, we enlarge the result obtained in [17] by studying the existence and uniqueness of solution for the following Atangana-Baleanu nonlinear fractional equation

$$\begin{cases} {}^{ABC}D^{\delta}z(t) = g(t, z(t)), \qquad t \in [0,1], \\ \\ {}^{ABC}D^{\delta-1}z(1) = 0; \quad {}^{ABC}D^{\delta-2}z(1) = 0; \quad z(0) = 0. \end{cases} \tag{1.1}$$

where ${}^{ABC}D^{\delta}$ is Atangana-Baleanu derivative in the sens of Caputo with the order $\delta \in (2,3]$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a function given.
The main advantage of this research is that it is not limited to studying the existence and uniqueness of solutions to nonlinear equations, but rather the application of the Atangana-Baleanu derivative to boundary conditions as an extension of it to the calculus fractional field and not only to the equation, which makes the area of fractional calculus rich in different derivative notions. Its application in contrast to previous works that were limited to other derivatives.
This work is divided into four Sections. The second is about the preliminaries, in this Section, we recall the necessary and some basic notion needed for fractional calculus, such as the gamma function, Betta, and Mittag-Leffler with one parameter, and with two parameters. It also include definitions and derivative properties of Atangana-Baleanu. In the third Section, we present our main results concerning the existence and uniqueness of solutions to fractional boundary value problems. We also provide an example to demonstrate our point in Section 4.

2. Preliminaries

In this part, we recall some necessary functions of fractional calculus. The essential definitions and characteristics of the fractional derivative proposed by Atangana and Baleanu [9, 18] are presented in this section, which will be used throughout the rest of this paper.

Definition 2.1. [21] It is well known that the classical Euler gamma function can be defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\ \mathrm{d}t.$$

Definition 2.2. [20] It is reasonable to assume that the Betta function can also be extended in a meaningful way. It is represented by an integral.

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\ \mathrm{d}x, \quad Re(s) > 0,\ Re(t) > 0.$$

Has a close relationship to gamma function

$$B(s,t)=B(t,s)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Definition 2.3. [11] the symbol for the fundamental Mittag-Leffler function is $E_\delta(z)$ and it is defined as

$$E_\delta(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\delta k)},\quad Re(\delta)>0.$$

One generalization of $E_\delta(z)$ is denoted and defined as follows

$$E_{\delta,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\delta k+\beta)},\quad Re(\delta)>0,\quad Re(\beta)>0,$$

which is a 2-parameter generalization of $E_\delta(z)$.

Definition 2.4. [9] Let $g\in H^1(a,b), b>a, \delta\in[0,1]$ then, the definition of Atangana– Baleanu derivative in Caputo type is given by:

$${}^{ABC}D_t^\delta\Big(g(t)\Big)=\frac{M(\delta)}{1-\delta}\int_a^t g'(x)E_\delta\left[-\delta\frac{(t-x)^\delta}{1-\delta}\right]dx,$$

where $M(\delta)$ denotes a normalization function obeying $M(0)=M(1)=1$.

Definition 2.5. [9] Let $g\in H^1(a,b), a<b, \delta\in[0,1]$, the Atangana-Baleanu derivative in Riemann-Liouville type is given by:

$${}_a^{ABR}D_t^\delta\Big[g(t)\Big]=\frac{M(\delta)}{1-\delta}\frac{d}{dt}\int_a^t g(x)E_\delta\left[-\delta\frac{(t-x)^\delta}{1-\delta}\right]dx.$$

Definition 2.6. The fractional integral related to the Atangana-Baleanu fractional derivative is defined by [9]:

$${}_a^{AB}I_t^\delta\Big[g(t)\Big]=\frac{1-\delta}{M(\delta)}g(t)+\frac{\delta}{M(\delta)\Gamma(\delta)}\int_a^t g(s)(t-s)^{\delta-1}ds.$$

Lemma 2.1. *[18] For $0<\delta<1$, we have*

$${}^{AB}I_a^\delta\Big({}^{ABC}D_a^\delta g(x)\Big)=g(x)-g(a),$$

and

$${}^{AB}I_b^\delta\Big({}^{ABC}D_b^\delta g(x)\Big)=g(x)-g(b).$$

Lemma 2.2. *[22] Let $n<\delta\le n+1$. Then ${}^{ABC}D^\delta g(t)=0$, if $g(t)$ is constant function.*

Definition 2.7. [22] Let $n<\delta\le n+1$ and g be function such that $g^{(n)}\in H^1(a,b)$. Set $\beta=\delta-n$. We define

$$\Big({}_a^{ABC}\mathbf{D}^\delta g\Big)(t)=\Big({}_a^{ABC}D^\beta g^{(n)}\Big)(t),$$

and it takes on the following form in the left Riemann-Liouville interpretation.

$$\left({}_{a}^{ABR}\mathbf{D}^{\delta}g\right)(t)=\left({}_{a}^{ABR}D^{\beta}g^{(n)}\right)(t).$$

We have the associated fractional integral

$$\left({}_{a}^{AB}\mathbf{I}^{\delta}g\right)(t)=\left(I^{n\,AB}I^{\beta}g\right)(t).$$

Proposition 2.1. *[22] For $u(t)$ defined on $[a,b]$ and $\delta\in(n,n+1]$, for some $n\in\mathbb{N}_0$, we have:*

- ${}^{ABR}\mathbf{D}^{\delta}\left({}^{AB}\mathbf{I}^{\delta}w(t)\right)=w(t)$.
- ${}_{a}^{AB}\mathbf{I}_{a}^{\delta}\left({}^{ABR}\mathbf{D}^{\delta}w(t)\right)=w(t)-\sum_{k=0}^{n-1}\frac{w^{(k)}(a)}{k!}(t-a)^k$.
- ${}_{a}^{AB}\mathbf{I}^{\delta}\left({}^{ABC}\mathbf{D}_{a}^{\delta}w(t)\right)=w(t)-\sum_{k=0}^{n}\frac{w^{(k)}(a)}{k!}(t-a)^k$.

3. Main result

In this section we are going to prove the existence and uniqueness of solution to the following boundary value problem

$$\begin{cases} {}^{ABC}D^{\delta}z(t)=g(t,z(t)), \qquad t\in[0,1], \\ \\ {}^{ABC}D^{\delta-1}z(1)=0;\quad {}^{ABC}D^{\delta-2}z(1)=0;\quad z(0)=0. \end{cases} \tag{3.1}$$

where $\delta\in(2,3]$ and we note that $I=[0,1]$ with given the condition of uniqueness. $C(I,\mathbb{R})$ means the continuous functions space from I into $\mathbb{R}$.
$\|z\|_\infty=\sup_{t\in[0,1]}|z(t)|$.

Theorem 3.1. *Let $2<\delta\leq 3$ and let $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. A function z is a solution of the initial value problem (3.1). If and only if z is a solution to the following integral equation*

$$\begin{aligned} z(t)&=P_1(t)\int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds \\ &+P_2(t)\int_0^1\int_0^s (s-\tau)^{\delta-3}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds \\ &-P_3(t)\int_0^1\int_0^s g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds \\ &-P_4(t)\int_0^1\int_0^s (s-\tau)^{\delta-2}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds \\ &+\frac{3-\delta}{M(\delta-2)}\int_0^t\int_0^s g(\tau,z(\tau))d\tau ds+\frac{\delta-2}{\Gamma(\delta)M(\delta-2)}\int_0^t (t-s)^{\delta-1}g(s,z(s))ds. \end{aligned} \tag{3.2}$$

Proof. Let as consider the following boundary value problem

$$\begin{cases} {}^{ABC}D^{\delta}z(t) = g(t, z(t)), \qquad t \in [0,1] \quad \delta \in (2,3], \\ \\ {}^{ABC}D^{\delta-1}z(1) = 0; \quad {}^{ABC}D^{\delta-2}z(1) = 0; \quad z(0) = 0. \end{cases}$$

We apply the fractional integral of Atangana-Baleanu in both sides of equation, we obtain

$$\begin{aligned} z(t) &= z(0) + tz'(0) + \frac{t^2}{2!}z''(0) + {}^{AB}I^{\delta-2}\left(\int_0^t \int_0^s g(\tau, z(\tau))d\tau ds\right) \\ z(t) &= z(0) + tz'(0) + \frac{t^2}{2!}z''(0) + \frac{1-(\delta-2)}{M(\delta-2)}\left(\int_0^t \int_0^s g(\tau, z(\tau))d\tau ds\right) \\ &+ \frac{\delta-2}{M(\delta-2)}\left({}^{RL}I^{(\delta-2)+2}g(t, z(t))\right) \\ z(t) &= z(0) + tz'(0) + \frac{t^2}{4}z''(0) + \frac{3-\delta}{M(\delta-2)}\left(\int_0^t \int_0^s g(\tau, z(\tau))d\tau ds\right) \\ &+ \frac{\delta-2}{M(\delta-2)}\left({}^{RL}I^{\delta}g(t, z(t))\right) \end{aligned}$$

We find the value of constants $z'(0)$ and $z''(0)$ by using the boundary condition. Or

$${}^{ABC}D^{\delta-1}z(1) = \left({}^{ABC}D^{\delta-2}z'(t)\right)\Big|_{t=1}$$

We calculate first $z'(t)$ then ${}^{ABC}D^{\delta-2}z'(t)$. So, we get

$$z'(t) = z'(0) + \frac{2}{4}tz''(0) + \frac{3-\delta}{M(\delta-2)}\int_0^t g(s, z(s))ds + \frac{\delta-2}{M(\delta-2)}{}^{RL}I^{\delta-1}g(t, z(t))$$

We apply the fractional derivative with order $\delta - 2$ and using definition (2.7), we obtain

$$\begin{aligned} {}^{ABC}D^{\delta-2}z'(t) &= \frac{M(\delta-2)}{2(3-\delta)}z''(0)tE_{\delta-2,2}\left(\frac{2-\delta}{3-\delta}t^{\delta-2}\right) \\ &+ \int_0^t g(s, z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(t-s)^{\delta-2}\right)ds \\ &+ \frac{\delta-2}{(3-\delta)\Gamma(\delta-2)}\int_0^t \int_0^s (s-\tau)^{\delta-3}g(\tau, z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(t-s)^{\delta-2}\right)ds. \end{aligned}$$

Since we have ${}^{ABC}D^{\delta-1}z(1) = 0$, then we get

$$\begin{aligned} {}^{ABC}D^{\delta-1}z(1) &= \frac{M(\delta-2)}{2(3-\delta)}z''(0)E_{\delta-2,2}\left(\frac{2-\delta}{3-\delta}\right) + \int_0^1 g(s, z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds \\ &+ \frac{\delta-2}{(3-\delta)\Gamma(\delta-2)}\int_0^1 \int_0^s (s-\tau)^{\delta-3}g(\tau, z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds = 0. \end{aligned}$$

By a simple calculation, we obtain the value of $z''(0)$, which is provided by

$$z''(0) = \frac{-2(3-\delta)}{M(\delta-2)E_{\delta-2,2}\left(\frac{2-\delta}{3-\delta}\right)} \int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds$$
$$- \frac{2(\delta-2)}{M(\delta-2)\Gamma(\delta-2)}$$
$$\times \frac{1}{E_{\delta-2,2}\left(\frac{2-\delta}{3-\delta}\right)} \int_0^1 \int_0^s (s-\tau)^{\delta-3} g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds.$$

We have, also

$$\begin{aligned} {}^{ABC}D^{\delta-2}z(t) &= \frac{M(\delta-2)}{3-\delta}z'(0)\int_0^t E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(t-s)^{\delta-2}\right)ds \\ &+ \frac{M(\delta-2)}{2(3-\delta)}z''(0)\int_0^t sE_{\delta-2}\left(\frac{2-\delta}{3-\delta}(t-s)^{\delta-2}\right)ds \\ &+ \int_0^t\int_0^s g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(t-s)^{\delta-2}\right)ds \\ &+ \frac{\delta-2}{(3-\delta)\Gamma(\delta-1)}\int_0^t\int_0^s (s-\tau)^{\delta-2}g(\tau,z(\tau)) \\ &\times E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(t-s)^{\delta-2}\right)d\tau ds. \end{aligned}$$

For $t=1$, ${}^{ABC}D^{\delta-2}z(1)=0$ we obtain the value of $z'(0)$, given by

$$\begin{aligned} z'(0) &= \frac{3-\delta}{M(\delta-2)}\int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds\frac{E_{\delta-2,3}(\frac{2-\delta}{3-\delta})}{E_{\delta-2,2}(\frac{2-\delta}{3-\delta})} \\ &+ \frac{(\delta-2)E_{\delta-2,3}(\frac{2-\delta}{3-\delta})}{\Gamma(\delta-2)M(\delta-2)E^2_{\delta-2,2}(\frac{2-\delta}{3-\delta})}\int_0^1\int_0^s (s-\tau)^{\delta-3}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds \\ &- \frac{3-\delta}{M(\delta-2)E_{\delta-2,2}}\int_0^1\int_0^s g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds \\ &- \frac{\delta-2}{\Gamma(\delta-1)M(\delta-2)E_{\delta-2,2}(\frac{2-\delta}{3-\delta})}\int_0^1\int_0^s (s-\tau)^{\delta-2}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds. \end{aligned}$$

Finally, we obtain the solution to equation (3.1), which is definable by

$$\begin{aligned} z(t) &= \left[\frac{3-\delta}{M(\delta-2)}\frac{E_{\delta-2,3}(\frac{2-\delta}{3-\delta})}{E^2_{\delta-2,2}(\frac{2-\delta}{3-\delta})}t - \frac{(3-\delta)t^2}{2M(\delta-2)E_{\delta-2,2}()}\right]\int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds \\ &+ \left[\frac{(\delta-2)}{M(\delta-2)\Gamma(\delta-2)}\frac{E_{\delta-2,3}(\frac{2-\delta}{3-\delta})}{E^2_{\delta-2,2}(\frac{2-\delta}{3-\delta})}t - \frac{(\delta-2)t^2}{2\Gamma(\delta-2)M(\delta-2)E_{\delta-2,2}(\frac{2-\delta}{3-\delta})}\right]\int_0^1\int_0^s (s-\tau)^{\delta-3}g(\tau,z(\tau)) \\ &\times E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds \\ &- \frac{(3-\delta)t}{E_{\delta-2,2}(\frac{2-\delta}{3-\delta})M(\delta-2)}\int_0^1\int_0^s g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds \end{aligned}$$

$$-\frac{(\delta-2)t}{M(\delta-2)E_{\delta-2,2}(\frac{2-\delta}{3-\delta})\Gamma(\delta-1)}\int_0^1\int_0^s(s-\tau)^{\delta-2}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$
$$+\frac{3-\delta}{M(\delta-2)}\int_0^t\int_0^s g(\tau,z(\tau))d\tau ds+\frac{\delta-2}{\Gamma(\delta)M(\delta-2)}\int_0^t(t-s)^{\delta-1}g(s,z(s))ds.$$

To simplify the solution formula, we used the following expression

$$P_1(t)=\frac{3-\delta}{M(\delta-2)}\frac{E_{\delta-2,3}(\frac{2-\delta}{3-\delta})}{E^2_{\delta-2,2}(\frac{2-\delta}{3-\delta})}t-\frac{(3-\delta)t^2}{2M(\delta-2)E_{\delta-2,2}()},$$
$$P_2(t)=\frac{(\delta-2)}{M(\delta-2)\Gamma(\delta-2)}\frac{E_{\delta-2,3}(\frac{2-\delta}{3-\delta})}{E^2_{\delta-2,2}(\frac{2-\delta}{3-\delta})}t-\frac{(\delta-2)t^2}{2\Gamma(\delta-2)M(\delta-2)E_{\delta-2,2}(\frac{2-\delta}{3-\delta})},$$
$$P_3(t)=\frac{(3-\delta)t}{E_{\delta-2,2}(\frac{2-\delta}{3-\delta})M(\delta-2)},$$
$$P_4(t)=\frac{(\delta-2)t}{M(\delta-2)\Gamma(\delta-1)E_{\delta-2,2}(\frac{2-\delta}{3-\delta})}.$$

Then the solution becomes

$$\begin{aligned}z(t)&=P_1(t)\int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds\\&+P_2(t)\int_0^1\int_0^s(s-\tau)^{\delta-3}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds\\&-P_3(t)\int_0^1\int_0^s g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds\\&-P_4(t)\int_0^1\int_0^s(s-\tau)^{\delta-2}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds\\&+\frac{3-\delta}{M(\delta-2)}\int_0^t\int_0^s g(\tau,z(\tau))d\tau ds+\frac{\delta-2}{\Gamma(\delta)M(\delta-2)}\int_0^t(t-s)^{\delta-1}g(s,z(s))ds.\end{aligned}$$

Suppose z satisfies the integral equation (3.2), and we proved that z verifies the fractional boundary value problem (3.1)

We have

$$\begin{aligned}z''(t)&=-\frac{3-\delta}{E_{\delta-2,2}(\frac{2-\delta}{3-\delta})M(\delta-2)}\int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds\\&-\frac{(\delta-2)}{\Gamma(\delta-2)M(\delta-2)E_{\delta-2,2}(\frac{2-\delta}{3-\delta})}\int_0^1\int_0^s(s-\tau)^{\delta-3}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds\\&+\frac{3-\delta}{M(\delta-2)}g(t,z(t))+\frac{\delta-2}{M(\delta-2)}{}^{RL}I^{\delta-2}g(t,z(t)).\end{aligned}$$

Then, we obtain

$${}^{ABC}D^{\delta-2}z''(t)={}^{ABC}D^{\delta-2}\left[-\frac{3-\delta}{E_{\delta-2,2}(\frac{2-\delta}{3-\delta})M(\delta-2)}\int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds\right]$$

$$-{}^{ABC}D^{\delta-2}\left[\frac{(\delta-2)}{\Gamma(\delta-2)M(\delta-2)E_{\delta-2,2}(\frac{2-\delta}{3-\delta})}\int_0^1\int_0^s(s-\tau)^{\delta-3}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds\right]$$

$$+{}^{ABC}D^{\delta-2}\left[{}^{AB}I^{\delta-2}g(t,z(t))\right]$$

So by lemma 2.2, we get

$${}^{ABC}D^{\delta}z(t)=g(t,z(t)).$$

Easily we verified that $z(0)=0$, Moreover, with a simple calculus, we obtain ${}^{ABC}D^{\delta-2}z(1)=0$ and ${}^{ABC}D^{\delta-1}z(1)=0$. □

Theorem 3.2. *Let $g: I\times\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function. Suppose that there exists a constant $k>0$, such that*

$$|g(t,z(t))-g(t,x(t))|<k\,|z(t)-x(t)|\,,\forall z,x\in\mathbb{R},\forall t\in I.$$

Then the problem (3.1) has a unique solution on I. If the following condition is satisfied

$$\frac{8k}{M(\delta-2)}<1. \tag{3.3}$$

Proof. The idea is to transform the problem (3.1) into a fixed point equation. For that, we consider the operator $T: C(I,\mathbb{R})\longrightarrow C(I,\mathbb{R})$ defined by

$$Tz(t)=P_1(t)\int_0^1 g(s,z(s))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds$$

$$+P_2(t)\int_0^1\int_0^s(s-\tau)^{\delta-3}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$-P_3(t)\int_0^1\int_0^s g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$-P_4(t)\int_0^1\int_0^s(s-\tau)^{\delta-2}g(\tau,z(\tau))E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$+\frac{3-\delta}{M(\delta-2)}\int_0^t\int_0^s g(\tau,z(\tau))d\tau ds+\frac{\delta-2}{\Gamma(\delta)M(\delta-2)}\int_0^t(t-s)^{\delta-1}g(s,z(s))ds.$$

Let $z_1,\ z_2 \in C(I,\mathbb{R})$ and $t\in I$

$$|Tz_2(t)-Tz_1(t)| \leq |P_1(t)|\int_0^1 |g(s,z_2(s))-g(s,z_1(s))|E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds$$

$$+|P_2(t)|\int_0^1\int_0^s (s-\tau)^{\delta-3}|g(\tau,z_2(\tau))-g(\tau,z_1(\tau))|E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$+|P_3(t)|\int_0^1\int_0^s |g(\tau,z_2(\tau))-g(\tau,z_1(\tau))|E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$+|P_4(t)|\int_0^1\int_0^s (s-\tau)^{\delta-2}|g(\tau,z_2(\tau))-g(\tau,z_1(\tau))|E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$+\frac{3-\delta}{M(\delta-2)}\int_0^t\int_0^s |g(\tau,z_2(\tau))-g(\tau,z_1(\tau))|d\tau ds$$

$$+\frac{\delta-2}{\Gamma(\delta)M(\delta-2)}\int_0^t (t-s)^{\delta-1}|g(s,z_2(s))-g(s,z_1(s))|ds$$

$$\leq |P_1(t)|k\left\|z_2-z_1\right\|_\infty\int_0^1 E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)ds$$

$$+|P_2(t)|k\left\|z_2-z_1\right\|_\infty\int_0^1\int_0^s (s-\tau)^{\delta-3}E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau$$

$$+|P_3(t)|k\left\|z_2-z_1\right\|_\infty\int_0^1\int_0^s E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$+|P_4(t)|k\left\|z_2-z_1\right\|_\infty\int_0^1\int_0^s (s-\tau)^{\delta-2}E_{\delta-2}\left(\frac{2-\delta}{3-\delta}(1-s)^{\delta-2}\right)d\tau ds$$

$$+\frac{3-\delta}{M(\delta-2)}k\left\|z_2-z_1\right\|_\infty\int_0^t\int_0^s d\tau ds+\frac{\delta-2}{\Gamma(\delta)M(\delta-2)}k\left\|z_2-z_1\right\|_\infty\int_0^t (t-s)^{\delta-1}ds$$

$$\leq \frac{2k}{M(\delta-2)\Gamma(\delta-2)}\left\|z_2-z_1\right\|_\infty+\frac{2k}{M(\delta-2)}\left\|z_2-z_1\right\|_\infty+\frac{1}{M(\delta-2)}k\left\|z_2-z_1\right\|_\infty$$

$$+\frac{1}{M(\delta-2)}k\left\|z_2-z_1\right\|_\infty+\frac{1}{M(\delta-2)}k\left\|z_2-z_1\right\|_\infty+\frac{1}{M(\delta-2)\Gamma(\delta)}k\left\|z_2-z_1\right\|_\infty$$

$$\leq \frac{8k}{M(\delta-2)}\left\|z_2-z_1\right\|_\infty.$$

Thus demonstrating that the operator T is a contraction mapping. The theorem's (3.2) implication is that T possesses a unique fixed point, which is a unique solution to the problem (3.1). □

4. Examples

In this illustration, we defend the truth of Theorem 3.2. We take into consideration the principal ABC fractional problem for $\delta = \frac{5}{2}$, and $g(t, z(t)) = \frac{5z(t)}{\exp(t)+127}$:

$$\begin{cases} {}^{ABC}D^{\frac{5}{2}} z(t) = \frac{5z(t)}{\exp(t)+127}, \\ \\ {}^{ABC}D^{\frac{3}{2}} z(1) = 0; \quad {}^{ABC}D^{\frac{1}{2}} z(1) = 0; \quad z(0) = 0. \end{cases} \tag{4.1}$$

It is clear that g is Lipschitz mapping with constant of lipschitz equal $k = \frac{5}{128}$. As for the condition (3.3), since $\frac{40}{128M(\frac{1}{2})} = 0,434 < 1$.

Thus, by the conclusion of Theorem 3.2, the boundary value problem (4.1) has a unique solution on $[0, 1]$.

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