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MUSIC AS MATHEMATICS OF SUBCONSCIOUSNES

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Abstract In this paper, we considered the structure and differences between the musical scales and the well-tempered chromatic scale, and the works of J.S. Bach. Also, we review further problems on a difference of musical scales and come to the subconscious feeling of excellent composers, including not only harmony but (dynamic) symmetry.

Keywords scales in music, continued fractions, quartic equation, analogue & digital music, golden ration, Fibonacci intervals

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1. Introduction

Our objective here is to exhibit **genuine musicians in their subconsciousness sense the touch and guiding of mathematical structure—notably symmetry** which shows up in composed music. Genuine musicians express their feeling of **harmonic structure** in their mind in terms of music driven by their architectural intuition of balance and unity.

In this paper we shall base our argument on the Helmholtz-Joachim scale §2 since this is in good match with just intonation and it is rarely found elsewhere than in [8]. It can serve as a cursor of scales in transient stage from Pythagorean, just intonation, Mercator scale to the rather mechanical equal temperament.

2. Helmholtz-Joachim scale

As have been talk about in [21], in the case of natural scales (just intonation) resp. Pythagorean scale, musical notes appear in the form $2^p3^q5^r$ (multiples of the basic note), where $p \in \mathbb{Z}$, $q = -3, -2, -1, 0, 1, 2, 3$ and $r = -1, 0, 1$, respectively $r = 0$.

H.von Helmholtz with the assist of the renowned violinist J. Joachim, made an experiment and tabulated the notes which are the most pleasing to the ears, which

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we mentioned to as the Helmholtz-Joachim scale [21]: HJ scale. It gave a partial description of Pythagoras' law of small numbers.

	C	$E^\flat$	E	F	G	$A^\flat$
int.	unison	minor 3rd	major 3rd	perfect 4th	perfect 5th	minor 6th
ratio	$\frac{1}{1}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$
note	do	$\mathrm{mi}^\flat$	mi	fa	sol	$\mathrm{la}^\flat$

Table 2.1. Helmholtz-Joachim scale

from C	A	C
interval	major sixth	octave
ratio	$\frac{5}{3}$	$\frac{2}{1}$
note	la	do

Table 2.1. Helmholtz-Joachim scale (cont.)

	C 264	D 297	E 330	F 352	G 396	A 440	B 495	C 528
C	$\times$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
D		$\times$	$\frac{10}{9}$	$\frac{32}{27}$	$\frac{4}{3}$	$\frac{40}{27}$	$\frac{15}{9}$	$\frac{16}{9}$
E			$\times$	$\frac{16}{15}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$
F				$\times$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$
G					$\times$	$\frac{10}{9}$	$\frac{5}{4}$	$\frac{4}{3}$
A						$\times$	$\frac{9}{8}$	$\frac{6}{5}$
B							$\times$	$\frac{16}{15}$

Table 2.2. Frequency ratios in just intonation scale

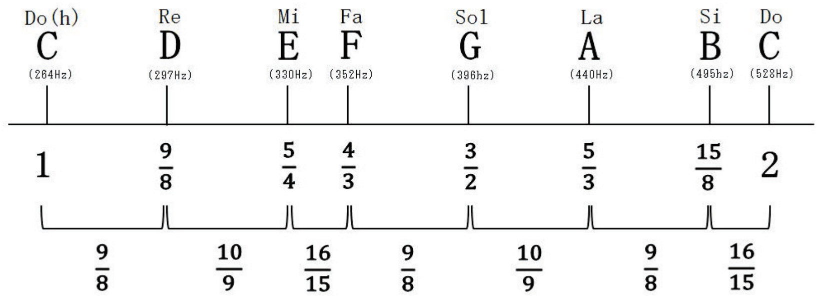

Figure 1. Just intonation scale

Comparing HJ scale and just intonation, we find a remarkable similarity. The only difference is D (re) $\frac{9}{8}$ is replaced by E$^\flat$ (mi$^\flat$) $\frac{6}{5}$ and instead of B (si) $\frac{15}{8}$, A$^\flat$ (la$^\flat$) $\frac{8}{5}$ is added.

A remarkable feature of HJ scale is that rach ratio except for the first and the last can be calculated just as with Farey fractions. E.g. to find G (sol) $\frac{3}{2}$ from the neighboring F (fa) $\frac{4}{3}$ A$^\flat$ (la$^\flat$ $\frac{8}{5}$) we calculate the mediant

$$\frac{4}{3} + \frac{8}{5} \overset{!}{=} \frac{4+8}{3+5} = \frac{3}{2}. \tag{2.1}$$

Farey fractions are constructed as follows: from $\frac{0}{1}, \frac{1}{1}$ we have $\frac{0}{1}, \frac{1}{2}, \frac{1}{1}$ from which $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$ and so on. The Farey sequence of order 2, 3. 4 are

$$F_2 = \left\{\frac{1}{2}, 1\right\}, \quad F_3 = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\} \quad F_4 = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}. \tag{2.2}$$

In general F_n is the set of all irreducible fractions with denominators $\leq n$.

Since a piano or an organ admits only 12 distinct notes in one octave, we must use the sma note for G$^\sharp$ and A$^\flat$. In the HJ scale, G$^\sharp$ is a major third $\frac{5}{4}$ (cf. Definition 3.1) above E (fa, so$^\sharp$), its distance from C is $\frac{5}{4} \cdot \frac{5}{4} = \frac{25}{16}$. Hence the distance from G$^\sharp$ to A$^\flat$ is not 1 as should be but

$$\frac{8}{5} \div \frac{25}{16} = \frac{128}{125} = 1.024,$$

which is perceptible to trained ears.

The whole tone $\frac{3}{2} \div \frac{4}{3} = \frac{9}{8}$ from F to G differs from the whole tone $\frac{10}{9}$ from E$\flat$ to F or from G to A ($\frac{5}{3} \div \frac{3}{2} = \frac{10}{9}$) by an amount that is called a comma by Greeks: $\frac{81}{80} = 1.0125$. This is to be compared with (2.4).

One octave has 12 **semitones**. Therefore, if we pile up the notes on the basic one, the 12th power is essential. In the case of the Pythagorean scale, what are piled up are powers of $\frac{3}{2}$, so that

$$\left(\frac{3}{2}\right)^{12} \approx 129.7, \tag{2.3}$$

which is a small higher than 7 octaves: $2^7 = 128$. The interval

$$\frac{129.7}{128} = 1.01338\cdots \tag{2.4}$$

is known as the **comma of Pythagoras**. This discrepancy accounts for some difficulties in attaining an organized system of pitch.

Helmholtz also observed that a sufficiently low note, say low C, has harmonics whose frequencies are exact multiples of the initial note (C). Table 1.2, an extract from the tabulation, is the foundation of Coxeter's speculation of the **law of cyclotomic numbers** [8]. Cf. §8 for Fibonacci intervals.

mult.	1	2	3	4	5	6	7	8	9	10	11	12
note	C	C	G	C	E	G	-	C	D	E	-	G

Table 2.3. Multiples of low C

3. From Pythagorean scale to just intonation

name	ratio	nat.	temp.	semi-tones
unison	$\frac{d}{d}$	$\frac{1}{1}$	1.0000	0
octave	$\frac{2d}{d}$	$\frac{2}{1}$	2.0000	12
perfect 5th	$\frac{s}{d} = \frac{l}{r} = \cdots = \frac{2m}{l} = \mathrm{q}$	$\frac{3}{2}$	1.4983	7
perfect 4th	$\frac{f}{d} = \frac{s}{r} = \cdots = \frac{2m}{t} = \mathrm{q}$	$\frac{4}{3}$	1.3348	5
major 3rd	$\frac{m}{d} = \frac{l}{f} = \frac{t}{s} = \frac{\mathrm{q}^4}{4}$	$\frac{5}{4}$	1.2599	4
minor 3rd	$\frac{f}{r} = \frac{s}{m} = \frac{2d}{l} = \frac{2r}{t} = \frac{4}{\mathrm{q}^3}$	$\frac{6}{5}$	1.1892	3
major 6th	$\frac{l}{d} = \frac{t}{r} = \frac{2r}{f} = \frac{2m}{s} = \frac{\mathrm{q}^3}{2}$	$\frac{5}{3}$	1.6818	9
minor 6th	$\frac{m}{d} = \frac{l}{f} = \frac{t}{s} = \frac{8}{\mathrm{q}^4}$	$\frac{8}{5}$	1.5874	8
major 2nd	$\frac{r}{d} = \frac{m}{r} = \frac{s}{f} = \frac{l}{s} = \frac{t}{l} = \frac{\mathrm{q}^2}{2}$	$\frac{9}{8}$	1.1225	2
minor 7th	$\frac{r}{d} = \frac{m}{r} = \frac{s}{f} = \frac{l}{s} = \frac{t}{l} = \frac{4}{\mathrm{q}^2}$	$\frac{16}{9}$	1.7818	10
minor 2nd	$\frac{f}{m} = \frac{2d}{t} = \frac{8}{\mathrm{q}^5}$	$\frac{16}{15}$	1.05946	1
major 7th	$\frac{2m}{f} = \frac{2d}{t} = \frac{\mathrm{q}^5}{4}$	$\frac{15}{8}$	1.8877	11
chromatic	$\frac{2m}{f} = \frac{2d}{t} = \frac{\mathrm{q}^5}{4}$	$\frac{25}{24}$	1.05946	1
aug. 4th	$\frac{t}{f} = \frac{\mathrm{q}^6}{8}$	$\frac{45}{32}$	1.4142	6
dim. 5th	$\frac{2f}{t} = \frac{16}{\mathrm{q}^6}$	$\frac{64}{45}$	1.4142	6
dim. 7th	$\frac{64}{\mathrm{q}^9}$	$\frac{128}{75}$	1.6818	9

Table 3.1. Full range of musical intervals

We explain part of Table 3.1 by the following

Definition 3.1. Two intervals which combine to give an octave is called an **inversion** to each other.

After the octave, the next simplest is the **perfect fifth** 3 : 2 containing 7 semitones whose inversion is the **perfect fourth** 4 : 3 containing 5 semitones. The

major third $5:4$ is the interval containing 4 semitones whose inversion is the **minor sixth** $8:5$ containing 8 semitones.

The sequence of three notes arranged in the order of the major third and the **minor third** $6:5$ is called a **major triad**. The minor third is the interval from E to G: $\frac{3}{2}\cdot\left(\frac{5}{4}\right)^{-1}=\frac{6}{5}$ which contains 3 semi-tones.

The common major chord (do-mi-so-do) has the ratio $4:5:6:8$, while the common minor chord (do-$mi^\flat$-so-do) has the ratio $10:12:15:20$.

There is a definition of the major triad and the minor triad seemingly different from that in Definition 3.1

Definition 3.2. The **major triad** is the superimposition of the major third by the perfect fifth on the root. The **minor triad** is the superimposition of the minor third by the perfect fifth on the root.

Proposition 3.1. *The major triad may be described as a sequence of* 4 *semi-tones followed by* 3 *semi-tones, altogether* 7 *semi-tones, or* $\frac{5}{4}\times\frac{6}{5}=\frac{3}{2}$*, the piling up of the major third followed by the minor third. which is the definition in Definition 3.1.*

This equality may also be expressed as $\frac{3}{2}\times\frac{5}{4}^{-1}=\frac{6}{5}$*, i.e. after superimposition of the major third by the perfect fifth on the root what is the interval following the major third? It is the minor 6th.*

In Corollary 3.2, it will be shown that there are exactly three major triads, $\{1,\frac{\mathrm{q}^4}{4},\mathrm{q}\}=\{d,m,s\}=$ CEG, $\{f,l,d\}=$ FAC, $\{s,t,r\}=$ GBD. Proposition 3.1 also determines them. Indeed, one octave may be thought of the sequence $\{2,2,1,2,2,2,1\}$ (of semi-tones) and then continues in the same way, so that the pattern of 4 semi-tones followed by 3 semi-tones is possible only for $2,2$ followed by $1,2$, which are CEG and = GBD or $2,2$ followed by $2,1$, which is = FAC.

The Pythagorean major 3rd $m=\frac{81}{64}$ is slightly bigger than the major 3rd $\frac{5}{4}$ of just intonation. The Pythagorean major 3rd is said to let the melody sound beautifully, but it diminishes harmony because of the beats contained. In $\{d,m\}$ the number of beats caused by the 4 times m and 5 times of d is

$$d\left(4\times\frac{m}{d}-5\times 1\right)=d\left(4\times\frac{81}{64}-5\times 1\right)=\frac{d}{16}=0.0625d, \tag{3.1}$$

which is 16.5 times$/s$ for $d=264$ Hz, say. Hence in the Pythagorean major triad

$$d\left\{1,\frac{m}{d},\frac{s}{d}\right\}=d\left\{1,\frac{\mathrm{q}^4}{4},\mathrm{q}\right\}=d\left\{1,\frac{81}{64},\frac{3}{2}\right\}, \tag{3.2}$$

there occur beats between the root and the major 3rd. To eliminate this beat, we decrease the major third by multiplying by the **syntonic comma**

$$\frac{1}{\Delta}=\frac{80}{81} \tag{3.3}$$

to make it $\mathbf{m}=\frac{5}{4}$. This yields the **just intonation**, in which the major 3rd consists of

$$\left\{1,\frac{\mathbf{m}}{d},\frac{s}{d}\right\}=\left\{1,\frac{\mathrm{q}^4\Delta}{4},\mathrm{q}\right\}=\left\{1,\frac{5}{4},\frac{3}{2}\right\}=\frac{1}{4}\left\{4,5,6\right\}. \tag{3.4}$$

For this, the value in (3.1) is 0 and there is no beat between the root and the major third.

	C 264	D 297	E 330	F 352	G 396	A 440	B 495	C 528
C	$\times$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
D		$\times$	$\frac{10}{9}$	$\frac{32}{27}$	$\frac{4}{3}$	$\frac{40}{27}$	$\frac{15}{9}$	$\frac{16}{9}$
E			$\times$	$\frac{16}{15}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$
F				$\times$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$
G					$\times$	$\frac{10}{9}$	$\frac{5}{4}$	$\frac{4}{3}$
A						$\times$	$\frac{9}{8}$	$\frac{6}{5}$
B							$\times$	$\frac{16}{15}$

Table 3.2. Frequency ratios in just intonation scale

Theorem 3.1. *The major chord in Table 3.1 is expressed as*

$$\begin{aligned} F &= \{d, r, m, f, s, l, t, 2d\} \\ &= d\Big\{1, \frac{\mathrm{q}^2}{2}, \frac{\mathrm{q}^4}{4}, \frac{2}{\mathrm{q}}, \mathrm{q}, \frac{\mathrm{q}^3}{2}, \frac{\mathrm{q}^5}{4}, 2\Big\} \end{aligned} \tag{3.5}$$

while the minor chord as

$$\begin{aligned} F &= \{l, t, d, r, m, f, s, 2l\} \\ &= l\Big\{1, \frac{\mathrm{q}^2}{2}, \frac{4}{\mathrm{q}^3}, \frac{2}{\mathrm{q}}, \mathrm{q}, \frac{8}{\mathrm{q}^4}, \frac{4}{\mathrm{q}^2}, 2\Big\}. \end{aligned} \tag{3.6}$$

Corollary 3.1. *In just intonation as well as in the Pythagorean scale* $\mathrm{q} = \frac{3}{2}$.

Corollary 3.2. *There are three major triads,* $\{1, \frac{\mathrm{q}^4}{4}, \mathrm{q}\} = \{d, m, s\} = \mathrm{CEG}$, $\{f, l, d\} = \mathrm{FAC}, \{s, t, r\} = \mathrm{GBD}$.

Proof. There are three major thirds $\frac{\mathrm{q}^4}{4} = \frac{m}{d}, \frac{l}{f}, \frac{t}{s}$ and so the piling up of minor thirds are possible only for those notes ending with m, l, t. $\square$

4. Continued fractions

Definition 4.1. For any $u \in \mathbb{R}\backslash\mathbb{Q}$, the following process is known as the **continued fraction expansion**.

(i) $u_0 = [u] \in \mathbb{Z}, \quad 0 < v_0 := u - u_0 < 1, \quad u = u_0 + v_0$

(ii) $u_1 - \left[\frac{1}{v_0}\right] \in \mathbb{N}, \quad 0 < v_1 = \frac{1}{v_0} - u_1 < 1, \quad v_0 = \frac{1}{u_1+v_1}$

$\cdots$

(n) $u_n - \left[\frac{1}{v_{n-1}}\right] \in \mathbb{N}, \quad 0 < v_n = \frac{1}{v_{n-1}} - u_n < 1, \quad u_{n-1} = \frac{1}{u_n+v_n}.$

$$u = u_0 + \frac{1}{u_1 + v_1} = u_0 + \frac{1}{u_1 + \frac{1}{u_2+v_2}} = u_0 + \frac{1}{u_1 + \frac{1}{u_2 + \frac{1}{\ddots + \frac{1}{u_n+v_n}}}}, \tag{4.1}$$
$$= [u_0; u_1, u_2, \cdots, u_n + v_n],$$

say. We refer to (4.1) as the nth continued fraction expansion of u. The process terminates for $u \in \mathbb{Q}$.

Lemma 4.1. *If we write* (4.1) *with* $v_n = 0$ *as*

$$\gamma_n = u_0 + \frac{1}{u_1 + \frac{1}{u_2 + \frac{1}{\ddots + \frac{1}{u_n}}}} = [u_0; u_1, u_2, \cdots, u_n] = \frac{x_n}{y_n}, \tag{4.2}$$

the nth **convergent***, then*

$$x_n = u_n x_{n-1} + x_{n-2}, \quad y_n = u_n y_{n-1} + y_{n-2}, \quad n \geq 3 \tag{4.3}$$

and

$$x_{n+1} y_n - x_n y_{n+1} = (-1)^n, \quad n \in \mathbb{N}. \tag{4.4}$$

Proof. First three values of convergents are as in the following table.

n	x_n	y_n
0	u_0	1
1	$u_0u_1 + 1$	u_1
2	$u_2(u_0u_1 + 1) + u_0$	$u_2u_1 + 1$

Table 4.1. Convergents

Proof of (4.3) is by induction. It is true for $n = 3$ by the table. Indeed, it is also true for $n = 2$. Assume it is true for n. Then γ_{n+1} is obtained from γ_n by replacing u_n by $u_n + \frac{1}{u_{n+1}}$. Hence

$$\gamma_{n+1} = \frac{\left(u_n + \frac{1}{u_{n+1}}\right) x_{n-1} + x_{n-2}}{\left(u_n + \frac{1}{u_{n+1}}\right) y_{n-1} + y_{n-2}} = \frac{x_{n+1}}{x_{n+1}}. \tag{4.5}$$

(4.4) follows from induction and (4.3). $\square$

Theorem 4.1. *The sequence* $\{\gamma_n\}$ *in* (4.2) *converges to* u *in* (4.1).

Proof. By the definition (4.2), replacing u_n by $u_n + v_n$, we should recover u:

$$u = \frac{(u_n + v_n)x_{n-1} + x_{n-2}}{(u_n + v_n)y_{n-1} + y_{n-2}} = \frac{y'_n}{x'_n}, \tag{4.6}$$

say. Hence

$$u - \gamma_n = \frac{x'_n y_n - x_n y'_n}{y_n y'_n} = \frac{v_n(x_{n-1}y_n - y_{n-1}x_n)}{y_n y'_n} = \frac{v_n(-1)^n}{y_n y'_n}$$

by (4.4). Since $0 < v_n < 1$ by Definition 4.1 and *a fortiori* $y_n < y'_n$, we infer that

$$|u - \gamma_n| < \frac{1}{y_n y'_n} < \frac{1}{y_n^2}. \tag{4.7}$$

Since $\{y_n\} \subset \mathbb{Z}$ is increasing and $y_n > 0$, we have $y_n \geq n$. Hence the inequaity (4.7) leads to $|u - \gamma_n| < \frac{1}{n^2} \to 0$, whence the result. □

Example 4.1. The golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ in [5] satisfies the quadratic equation

$$\phi^2 = \phi + 1 \tag{4.8}$$

or

$$\phi = 1 + \frac{1}{\phi}.$$

Hence apparently,

$$\phi = [1; 1, 1, \cdots] = [1.\bar{1}].$$

(4.8) is a characteristic equation for the Fibonacci sequence $\{F_n\}$ satisfying the recurrence $F_{n+1} = F_n +_{n-1}$ with $F_0 = F_1 = 1$, cf. §8.

5. Mercator 53-note tempered scale

Nicholas Mercator (1620-1687) proposed a 53-note tempered scale. Coxeter [8, p.318] speculates the reason why 53 appears on the basis of continued fractions. This choice of the number 53 seems to be made by the following reasoning. To find an appropriate solution to the equation

$$2^x = \frac{3}{2} \tag{5.1}$$

or $2^{x+1} = 3$, which is equivalent to

$$(x + 1) \log 2 = \log 3, \tag{5.2}$$

or $x = \frac{\log 3}{\log 2} - 1$.

To find the continued fraction expansion of $\frac{\log 3}{\log 2}$, he uses the approximation of the logarithms $\log 2 = 0.3010300$, $\log 3 = 0.4771213$ and finds the expansion of $\frac{1760913}{3010300}$ which is $\frac{\log 3}{\log 2} - 1$.

$$u_0 + \frac{1}{u_1+}\frac{1}{u_2+}\frac{1}{u_3+}\cdots\frac{1}{u_n} = u_0 + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{u_3 + \cfrac{1}{\ddots + \cfrac{1}{u_n}}}}} \tag{5.3}$$

$$\frac{1760913}{3010300} = \frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{5+}\frac{1}{2+}\frac{1}{23+}\cdots = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots+}}}} \tag{5.4}$$

The fourth convergent is

$$\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\frac{1}{2}}}} = \frac{7}{12}. \tag{5.5}$$

This is the 12-note tempered scale.

The sixth convergent is

$$\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{3+1} = \frac{31}{53}. \tag{5.6}$$

This is the 53-note tempered scale due to Mercator.

This use of the number 53 is implied by the Pythagorean division of the tone and semitone into 9 parts and 4 parts, respectively. Since the diatonic scale consists of 5 tones and 2 semitones, it follows that the octave contains

$$9+9+4+9+9+9+4 = 53 \tag{5.7}$$

of these parts, and the perfect fifth (3 tones and 1 semitone) contains

$$9+9+4+9 = 31. \tag{5.8}$$

To deal with just intonation we consider the equation

$$2^x = \left(\frac{3}{2}\right)^p \left(\frac{5}{4}\right)^q \tag{5.9}$$

in x in the form $x = [x] + \varepsilon$ with arbitrarily small ε. This leads to a problem in Diophantine approximation and we hope to return to this elsewhere.

6. Bach's fifth

Remember Corollary 3.2. We consider Kellner's tuning. In the major triad $\{1, \frac{\mathrm{q}^4}{4}\} = \{d, m, s\}, \{f, l, d\}, \{s, t, r\}$, we make q a little smaller than natural 5th so that the number of beats caused by three times the root and twice the 5th note be equal to that of the beat caused by the four times the the 3rd and the 5th times of the root, i.e. writing such a q by $\mathrm{q_B}$, we have

$$3\times 1 - 2\times \mathrm{q_B} = 4\times\frac{\mathrm{q_B^4}}{4} - 5\times 1$$

whence we are move to the quartic equation

$$\mathrm{q_B^4} + 2\mathrm{q_B} - 8 = 0 \tag{6.1}$$

called **Bach's equation**. Numerically

$$\mathrm{q_B} = 1.4959535062432299\cdots. \tag{6.2}$$

6.1. Solution of a quartic equation

In this subsection, we give a most reachable technique of solving (6.2) than that given in [21]. We shall obtain the (real) roots of a quartic equation

$$W^4 + a_1W^2 + b_1W + c_1 \ = \ 0$$

eventually with $a_1 = 0, b_1 = 2, c_1 = -8$. This can be reduced to a cubic equation by using Ferrari's method. We add $2kW^2+k^2$ to both sides of $W^4 = -a_1W^2-b_1W-c_1$ to deduce that

$$(W^2 + k)^2 = \ (2k - a_1)W^2 - b_1W + k^2 - c_1. \tag{6.3}$$

We choose the parameter k so that the right-hand side is a square of a linear polynomial, i.e. the discriminant is equal to zero.

$$\begin{aligned} D = & \ b_1^2 - 4(2k - a_1)(k^2 - c_1) \\ = & \ -8k^3 + 4a_1k^2 + 8c_1k - 4a_1c_1 + b_1^2 = 0 \end{aligned} \tag{6.4}$$

With $Y = 2k$, (6.4) amounts to

$$Y^3 - a_1Y^2 - 4c_1Y + 4a_1c_1 - b_1^2 = \ Y^3 + pY + q \ = \ 0, \tag{6.5}$$

say (in the case $a_1 = 0$). This, being a cubic equation, can be solved by the method of Cardano. We use a variant of the Lagrange resolvent and determine u, v $(u \le v)$ satisfying (when $p = 32, q = -4$)

$$\begin{aligned} Y^3 + pY + q = Y^3 + 32Y - 4 &= Y^3 + u^3 + v^3 - 3uvY \\ &= (Y + u + v)(Y^2 + u^2 + v^2 - uY - vY - uv). \end{aligned} \tag{6.6}$$

(6.6) implies

$$\begin{cases} -3uv & = p \\ u^3 + v^3 & = q \end{cases} \tag{6.7}$$

Since $u^3v^3 = -\frac{p^3}{27} = \frac{32^3}{27}$, it follows that u^3, v^3 are the roots of the equation

$$Z^2 - qZ - \frac{p^3}{27} = 0. \tag{6.8}$$

Hence

$$Z \ = \ \frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \ = \ -2 \pm 2\sqrt{\frac{8219}{27}} = -2 \pm 34.8945501422447 \tag{6.9}$$

and so

$$u \ = \ -\left(2 + 2\sqrt{\frac{8219}{27}}\right)^{\frac{1}{3}} = -3.3290532330643168$$

$$v = \left(-2+2\sqrt{\frac{8219}{27}}\right)^{\frac{1}{3}} = 3.2041141789878327.$$

Since the real root must be $Y = -u - v$ from (6.6), we have

$$2k = Y = -u - v = \left(2+2\sqrt{\frac{8219}{27}}\right)^{\frac{1}{3}} - \left(-2+2\sqrt{\frac{8219}{27}}\right)^{\frac{1}{3}} = 0.12493905407648409 \tag{6.10}$$

$2k$ satisfies (6.4)Then (6.3)with $a_1 = 0, b_1 = 2$ reads

$$(W^2+k)^2 = \sqrt{2k}\left(W - \frac{1}{2k}\right)^2$$

which leads to $W^2 \pm \left(\sqrt{2k}W - \frac{1}{\sqrt{2k}}\right) + k = 0$. Of them only

$$W^2 + \sqrt{2k}W - \frac{1}{\sqrt{2k}} + k = 0$$

has real roots which are

$$W = -\frac{\sqrt{2k}}{2} \pm \frac{1}{2}\sqrt{\frac{4}{\sqrt{2k}} - 2k} = 1.4959535062432299, -1.8494206957764086, \tag{6.11}$$

the positive root leading to (6.2).

7. Analogue vs. Digital music

It is often said, especially by audience which has trained musical ears, that music played by digital devices sound rather **flat** and even boring compared with live performance. There is a good reason for that. In transforming analogue signal into digital signal, the A/D transformer being used, which makes quantization and approximates the sampled signal at digital levels. This **rounding-off** of analogue signals give them a sort of **unpleasant perfection**. A good example is a karaoke estimater which would give bad marks for those professional singers who can get the audience touched, on the ground that they don't sing as in the scores.

7.1. HiFi vs. WiFi

In [19, pp.30-33] the principle of CD is expressed. The pitch of the sound can be divided into $2^{16} = 65536$ parts because a CD can record 16 convex-concave points as one information and transforms into 0 and 1 signal. The CD reads these 16 information 44100 times per second. The cause for this depends on the supposition that **human ears can hear the sound whose frequencies are up to** $20kHz = 20000Hz$. Since the sound with frequency $20000Hz$ oscillates 20000 times per second and so more than this times of sampling is needed. And for stereo recording, we need twice as numerous, whence the sampling frequency 44100. Thus the sampling (Nyquist) rate $\frac{\pi}{T}$ is 1/44100 and the input digital signal is re-establish by the sampling theorem [6] etc.

Thus digital art smells of death.

We interpret this as the contrast between the pentagon (human body with arms held horizontal) and the hexagon (snow crystal).

8. The golden ratio and Fibonacci intervals

The home key could be any one of seven notes but what survived in tone-centered music nowadays are the **Ionian scale** beginning and ending on C and the **Aeolian scale** (with key-note A). The Ionian and Aeolian scales are known as the ordinary major and minor scales. Pursuing the reason why these survived of all other possible 6 modes, we encountered the speculation in [4].

	begin	key-note
Ionian	C	major
Aeolian	A	minor

Table 8.1. Ionian vs. Aeolian scale

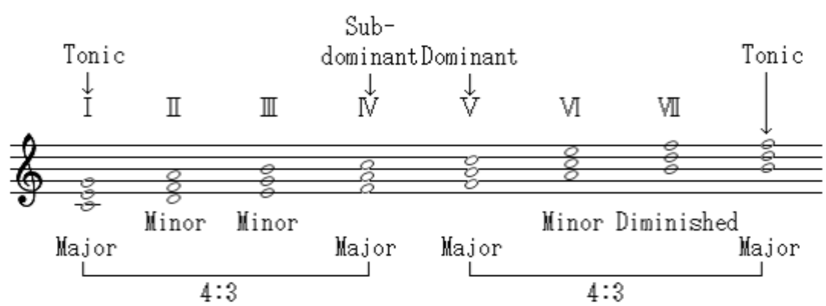

Figure 2. The position of major triads

Definition 8.1. The **major triad** is the piling up of the major third followed by the perfect fifth on the root. The **minor triad** is the piling up of the minor third followed by the perfect fifth on the root.

In paper [4], there are two speculations for the cause why those two chords remain. One is that the interval from si to do is tiny and gives the impression that it is coming to an end. The second one is more convincing that as one can see, the major triads are situated symmetrically over the octave, and so these chords survived. But although in the figure it is apparent, it is not certain whether one senses this symmetry by ears. We estimate that this is not a reasoning by senses but by causes. In paper [4], it is declared that Bach's music is quite mathematical but for Bach, music comes first and the accompanying mathematical structure is rather the by-products of composing music—as mathematics of subconsciousness or senses.

Figure 3. Hokusai: Mt. Fuji at the back of waves

Fibonacci intervals (counting in semitones) in Bartók's Sonata for Two Pianos and Percussion, 3rd mov. (1937) (Maconie 2005, 26, 28, [20])—Wikipedia.

We note that only first 4 Fibonacci numbers are used, $1, 2, 3, 5, 8$ which are stated as giving harmonics by Helmholtz, Table 1.2.

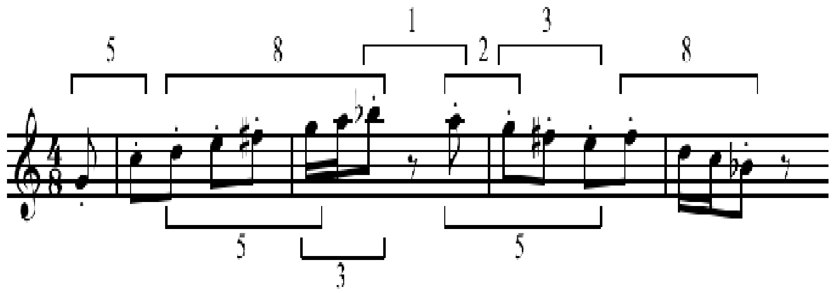

Figure 4. Bartok: Fibonacci

8.1. Quellenangaben

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References on matheasthetics: [7], [10], [11], [12], [13], [14], [15], [27].
References on fluctuations: [9], [17], [18], [22], [26], [28].

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