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PARAMETRIC INTEGRAL TRANSFORMS AND THEIR APPLICATIONS*

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Abstract In this study, we present parametric variations of popular integral transforms, such as the Laplace transform and Stieltjes transform. We demonstrate iteration identities and Parseval-Goldstein-like relationships that involve these parametric integral transforms. Moreover, we utilize these findings to compute improper integrals of established functions, such as the MacDonald function and the Tricomi function.

Keywords Parametric Laplace transforms, Parametric Stieltjes transforms, Parametric Fourier sine and cosine transforms, Mellin transforms.

MSC(2010) 44A10, 44A15.

1. Introduction

Goldstein [6] introduced the relationship

$$\int_0^\infty f(x)\,\mathscr{L}\big[g(y)\big](x)\,dx = \int_0^\infty g(y)\,\mathscr{L}\big[f(x)\big](y)\,dy. \tag{1.1}$$

for the classical Laplace transform

$$\mathscr{L}\big[f(x)\big](y) = \int_0^\infty e^{-xy}\,f(x)\,dx. \tag{1.2}$$

Goldstein used the relation Equation (1.1) some representation for Whittaker's confluent hypergeometric function. The relation Equation (1.1) is called an exchange identity by Vander pol and Bremmar [14] and the relation is referred as the Goldstein exchange identity. Similar relationships were obtained for other integral transforms such as the Hardy transform by Srivastava [12], the potential transform by Srivastava and Singh [13], and the generalized Hankel transform by Agarwal [1]. Readers may benefit from exploring the related topics discussed in two articles authored by Choi and Agarwal [2,3].

It is known that the Stieltjes transform:

$$\mathscr{S}\big[f(x)\big](y) = \int_0^\infty \frac{f(x)}{x+y}\,dx \tag{1.3}$$

arises naturally as an iteration of the Laplace transform:

$$\mathscr{L}\Big[\mathscr{L}\big[f(x)\big](u)\Big](y) = \mathscr{L}^2\big[f(x)\big](y) = \mathscr{S}\big[f(x)\big](y) \tag{1.4}$$

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The iteration identity Equation (1.4) can be used to calculate and invert Stieltjes transforms, for example, see Widder [15] and Sneddon [11].

It is also shown that the generalization of the Stieltjes transform:

$$\mathscr{S}_{\rho}\big[f(x)\big](y) = \int_0^\infty \frac{f(x)}{(x+y)^\rho}\,dx \tag{1.5}$$

is the iteration of Laplace transforms:

$$\mathscr{L}\Big[u^{\rho-1}\mathscr{L}\big[f(x)\big](u)\Big](y) = \Gamma(\rho)\mathscr{S}_{\rho}\big[f(x)\big](y), \tag{1.6}$$

(cf. [10, Equation (4.16), p. 86]).

In this paper, we introduce the parametric Laplace transform

$$\mathscr{L}_{\lambda}\big[f(x)\big](y) = \int_0^\infty x^{\lambda-1}e^{-xy}f(x)\,dx \tag{1.7}$$

and the parametric Stieltjes transform

$$\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y) = \int_0^\infty \frac{x^{\lambda_1-1}f(x)}{(x+y)^{\lambda_2}}\,dx. \tag{1.8}$$

We obtain a Parseval-Goldstein-type relationship for the parametric Laplace transform and the parametric generalized Stieltjes transform. Furthermore, we obtain new relationships for the parametric Fourier sine transform

$$\mathscr{F}_{s,\lambda}\big[f(x)\big](y) = \int_0^\infty x^{\lambda-1}\sin(xy)f(x)\,dx. \tag{1.9}$$

and the parametric Fourier cosine transform

$$\mathscr{F}_{c,\lambda}\big[f(x)\big](y) = \int_0^\infty x^{\lambda-1}\cos(xy)f(x)\,dx. \tag{1.10}$$

We also establish relationships for the Mellin transform:

$$\mathscr{M}\big[f(x)\big](y) = \int_0^\infty x^{y-1}f(x)\,dx. \tag{1.11}$$

We note that if $\lambda_1 \neq \lambda_2$, then

$$\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y) \neq \mathscr{S}_{\lambda_2,\lambda_1}\big[f(x)\big](y). \tag{1.12}$$

If $\lambda_1 = \lambda_2 = \lambda$, then

$$\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y) = \mathscr{S}_{\lambda}\big[x^{\lambda-1}f(x)\big](y). \tag{1.13}$$

2. Iteration Identities

We start with the iteration identity that establishes a relationship between the parametric Laplace transform (1.7) and the parametric Stieltjes transform (1.3).

Lemma 2.1. *If* $\Re(\lambda_1) > 0$ *and* $\Re(\lambda_2) > 0$*, then*

$$\mathscr{L}_{\lambda_1}\Big[\mathscr{L}_{\lambda_2}\big[f(x)\big](u)\Big](y) = \Gamma(\lambda_1)\mathscr{S}_{\lambda_2,\lambda_1}\big[f(x)\big](y) \tag{2.1}$$

$$\begin{aligned}\mathscr{L}_{\lambda_1}\Big[\mathscr{L}_{\lambda_2}\Big[\mathscr{L}_{\lambda_3}\big[f(x)\big](u)\Big](v)\Big](y) &= \Gamma(\lambda_2)\mathscr{L}_{\lambda_1}\Big[\mathscr{S}_{\lambda_3,\lambda_2}\big[f(x)\big](u)\Big](y) \\ &= \Gamma(\lambda_1)\mathscr{S}_{\lambda_2,\lambda_1}\Big[\mathscr{L}_{\lambda_3}\big[f(x)\big](u)\Big](y) \end{aligned} \tag{2.2}$$

Proof. If we iterate the the generalized Laplace transform, we have the relation

$$\mathscr{L}_{\lambda_1}\big[\mathscr{L}_{\lambda_2}\big[f(x)\big](u)\big](y) = \int_0^\infty u^{\lambda_1-1}e^{-uy}\left[\int_0^\infty x^{\lambda_2-1}e^{-xu}f(x)\,dx\right]du. \tag{2.3}$$

Changing the order of integration and using the definition of the Laplace transform (1.2), we see that

$$\begin{aligned}\mathscr{L}_{\lambda_1}\big[\mathscr{L}_{\lambda_2}\big[f(x)\big](u)\big](y) &= \int_0^\infty x^{\lambda_2-1}f(x)\left[\int_0^\infty u^{\lambda_1-1}e^{-(x+y)u}\,du\right]dx \\ &= \int_0^\infty x^{\lambda_2-1}f(x)\mathscr{L}\big[u^{\lambda_1-1}\big](x+y)\,dx \end{aligned} \tag{2.4}$$

Computing the inner integral on the right-hand side and using the definition of the parametric Stieltjes transform Equation (1.8), we deduce the iteration identity Equation (2.1).

The proof of iteration identity Equation (2.2) follow from the identity Equation (2.1). □

Remark 2.1. The iteration identity in Equation (2.1) generalizes the iteration identity Equation (1.4).

As an illustration of the iteration identity Equation (2.1) of Lemma (2.1), we present the identities for the parametric Stieltjes transform, the Beta function

$$B(\nu,\mu) = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)} = \int_0^\infty t^{\nu-1}(1-t)^{\mu-1}\,dt \tag{2.5}$$

(cf. [9, p. 603, (58:1:1)]), and the integral definition of Tricomi function

$$U(a,c,x) = \frac{1}{\Gamma(a)}\int_0^\infty \frac{t^{a-1}}{(1+t)^{1+a-c}}\,dt \tag{2.6}$$

(cf. [9, p. 499, (48:3:6)]) in the following theorem.

Theorem 2.1. *If* $\Re(\lambda_1) > \Re(\lambda_1-\lambda_2) > 0$*, then we have*

$$\mathscr{S}_{\lambda_1,\lambda_2}\left[x^\mu\right](y) = y^{\lambda_1+\mu-\lambda_2}\mathrm{B}(\lambda_1+\mu,\lambda_2-\lambda_1-\mu) \tag{2.7}$$

and

$$\mathscr{S}_{\lambda_1,\lambda_2}\left[x^\mu e^{-\beta x}\right](y) = \Gamma(\lambda_1+\mu)y^{\lambda_1-\lambda_2+\mu}\mathrm{U}(\lambda_1+\mu,1+\lambda_1-\lambda_2+\mu,\beta y) \tag{2.8}$$

Proof. In the iteration identity Equation (2.1) of Lemma (2.1) we set

$$f(x) = x^\mu \tag{2.9}$$

so that

$$\mathscr{L}_{\lambda_1}\left[x^{\mu}\right](u)=\frac{\Gamma(\lambda_1+\mu)}{u^{\lambda_1+\mu}} \quad \text{and} \quad \mathscr{L}_{\lambda_2}\left[u^{-\lambda_1-\mu}\right](y)=\frac{\Gamma(\lambda_2-\lambda_1-\mu)}{y^{\lambda_2-\lambda_1-\mu}}\cdot \tag{2.10}$$

Using the iteration identity Equation (2.1) and the results Equation (2.10) we obtain the assertion (2.7).

Similary, we set

$$f(x)=x^{\mu}e^{-\beta x} \tag{2.11}$$

in the iteration identity Equation (2.1) of Lemma (2.1) and using the definition (2.6) of the Tricomi function

$$\mathrm{U}(a,1+a-b,\beta u)=\frac{1}{\Gamma(a)}u^{b-a}\int_0^{\infty}\frac{x^{a-1}e^{-\beta x}}{(x+u)^b}\,dx \tag{2.12}$$

we obtain the assertion Equation (2.8). □

The following corollary establishes a useful relationship between the beta and the Tricomi function.

Corollary 2.1. *We have*

$$B(z_1,z_2)=\Gamma(z_1)U(z_1,1-z_2,0) \tag{2.13}$$

Proof. In the identity (2.8) of Theorem 2.1 put $\beta=0$, then we have

$$B(\lambda_1+\mu,\lambda_2-\lambda_1-\mu)=\Gamma(\lambda_1+\mu)\mathrm{U}(\lambda_1+\mu,\lambda_1-\lambda_2+\mu+1)$$

The identity (2.13) is obtained by substituting $\lambda_1+\mu=z_1$ and $\lambda_2-\lambda_1-\mu=z_2$.

This can also be simply proven by using definitions of the beta function (2.5) and the Tricomi function (2.6). □

3. Parseval-Goldstein Relationships

We start with a Parseval-Goldstein relationship between the parametric Laplace transform Equation (1.7) and the parametric Stieltjes transform (1.8).

Theorem 3.1. *If* $\Re(\lambda_1)>0$ *and* $\Re(\lambda_2)>0$*, then*

$$\int_0^{\infty}u^{\lambda_2-1}\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\mathscr{L}_{\lambda_2}\big[g(y)\big](u)\,du = \Gamma(\lambda_2)\int_0^{\infty}y^{\lambda_2-1}g(y)\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y)\,dy \tag{3.1}$$

$$\int_0^{\infty}u^{\lambda_1-1}\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\mathscr{L}_{\lambda_2}\big[g(y)\big](u)\,du = \Gamma(\lambda_1)\int_0^{\infty}x^{\lambda_1-1}f(x)\mathscr{S}_{\lambda_2,\lambda_1}\big[g(y)\big](x)\,dx \tag{3.2}$$

Proof. Using the definition of the parametric Laplace transform Equation (1.2) we have

$$\int_0^{\infty}u^{\lambda_2-1}\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\mathscr{L}_{\lambda_2}\big[g(y)\big](u)\,du$$

$$= \int_0^\infty u^{\lambda_2-1}\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\left[\int_0^\infty y^{\lambda_2-1}e^{-uy}g(y)\,dy\right]du. \tag{3.3}$$

Changing the order of integration, which is permissible by the absolute convergence of the integrals, we find from Equation (3.3) that

$$\begin{aligned}\int_0^\infty & u^{\lambda_2-1}\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\mathscr{L}_{\lambda_2}\big[g(y)\big](u)\,du\\ &= \int_0^\infty y^{\lambda_2-1}g(y)\left[\int_0^\infty u^{\lambda_2-1}e^{-uy}\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\,du\right]dy\\ &= \int_0^\infty y^{\lambda_2-1}g(y)\mathscr{L}_{\lambda_2}\Big[\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\Big](y)\,dy. \end{aligned}\tag{3.4}$$

Now the assertion Equation (3.1) follows from the iteration identity Equation (2.1) and the definition of the parametric Stieltjes transform (1.8).

The proof of the second assertion Equation (3.2) is similar. □

An immediate corollary of Theorem 3.1 is the following:

Corollary 3.1. *We have*

$$\mathscr{F}_{s,\lambda_2}\big[\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y)\big](z) = \int_0^\infty \frac{u^{\lambda_2-1}}{(u^2+z^2)^{\lambda_2/2}}\sin\left[\lambda_2\arctan\left(\frac{z}{u}\right)\right]\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\,du \tag{3.5}$$

$$\mathscr{F}_{c,\lambda_2}\big[\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y)\big](z) = \int_0^\infty \frac{u^{\lambda_2-1}}{(u^2+z^2)^{\lambda_2/2}}\cos\left[\lambda_2\arctan\left(\frac{z}{u}\right)\right]\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\,du \tag{3.6}$$

Proof. In the relationship Equation (3.1) of Theorem 3.1 we set

$$g(y) = \sin(zy) \tag{3.7}$$

so that

$$\mathscr{L}_{\lambda_2}\left[\sin(zy)\right](u) = \mathscr{L}\left[y^{\lambda_2-1}\sin(zy)\right](u) = \frac{\Gamma(\lambda_2)}{(u^2+z^2)^{\lambda_2/2}}\sin\left[\lambda_2\left(\frac{z}{u}\right)\right], \tag{3.8}$$

where we used the result [4, p. 152, 4.7 (15)]. Substituting the results Equation (3.7) and Equation (3.8) we obtain the assertion Equation (3.5).

Similary, we set

$$g(y) = \cos(zy) \tag{3.9}$$

in the relationship Equation (3.1) of Theorem 3.1 and using the result Equation [4, p. 157, 4.7 (58)], we obtain the assertion Equation (3.6). □

Another corollary of Theorem 3.1 is:

Corollary 3.2. *We have*

$$\mathscr{S}_{\lambda_2,\lambda_2}\big[\mathscr{L}_{\lambda_1}\big[f(x)\big](u)\big](z) = \mathscr{L}_{\lambda_2}\big[\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y)\big](z) \tag{3.10}$$

$$\mathscr{S}_{\lambda_2,\lambda_1}\big[\mathscr{L}_{\lambda_2}\big[g(y)\big](u)\big](z) = \Gamma(\lambda_2)\int_0^\infty y^{\lambda_1-1}\mathrm{U}(\lambda_1, 1+\lambda_1-\lambda_2, zy)g(y)\,dy. \tag{3.11}$$

Proof. In the relationship Equation (3.1) of Theorem 3.1 we set

$$g(y) = e^{-zy}. \tag{3.12}$$

The assertion immediately follows upon using the definitions of Equations (1.7), (1.8), and (3.1).

Similary, we set

$$f(x) = e^{-zx} \tag{3.13}$$

in the relationship Equation (3.1) of Theorem 3.1 and using the result Equation (2.12), we obtain the assertion Equation (3.11). □

The following corollary establishes a useful relationship between the Tricomi function and the hypergeometric function:

$${}_2F_1\left(a,b;c;z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{3.14}$$

where the symbol $(a)_n$ is defined as

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}\cdot \tag{3.15}$$

Corollary 3.3. *We have*

$$\begin{aligned}\int_0^\infty y^{\lambda_1-1} g(y) {}_2F_1\left(a,\lambda_1;\mu+\lambda_2;1-\frac{y}{z}\right) dy &= \frac{\Gamma(\mu+\lambda_2) z^{\lambda_1}}{\Gamma(\mu-\lambda_1+\lambda_2)\Gamma(\lambda_2)} \\ &\times \int_0^\infty u^{\lambda_2-1} \mathrm{U}\left(\lambda_1, 1+\lambda_1-\mu; uz\right) \mathscr{L}_{\lambda_2}\big[g(y)\big](u)\, du \end{aligned} \tag{3.16}$$

$$\begin{aligned}\mathscr{S}_{\lambda_2,\mu}\big[\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y)\big](z) &= z^{\lambda_2-\mu} \\ &\times \int_0^\infty u^{\lambda_2-1} \mathscr{L}_{\lambda_1}\big[f(x)\big](u) \mathrm{U}\left(\lambda_2, 1+\lambda_2-\mu, uz\right) du \end{aligned} \tag{3.17}$$

Proof. In the relationship Equation (3.1) of Theorem 3.1 we set

$$f(x) = \frac{1}{(x+z)^{\mu}}\cdot \tag{3.18}$$

Using the result Equation (2.12), we have

$$\mathscr{L}_{\lambda_1}\big[f(x)\big](u) = \Gamma(\lambda_1) z^{\lambda_1-\mu} \mathrm{U}\left(\lambda_1, 1+\lambda_1-\mu, uz\right), \tag{3.19}$$

and the result [5, p. 233, 4.4 (9)] we have

$$\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y) = \frac{\Gamma(\lambda_1)\Gamma(\mu-\lambda_1+\lambda_2)}{\Gamma(\mu+\lambda_2)} \frac{y^{\lambda_1-\lambda_2}}{z^{\mu}} {}_2\mathrm{F}_1\left(\mu,\lambda_1;\mu+\lambda_2;1-\frac{y}{z}\right) \tag{3.20}$$

Substituting the results Equations (3.18), (3.19), and (3.20), we obtain the assertion (3.16).

In the relationship Equation (3.1) of Theorem 3.1 we set

$$g(y) = \frac{1}{(y+z)^{\mu}}\cdot \tag{3.21}$$

Using the identity Equation (2.12) and substituting the results to the relationship Equation (3.1), we obtain the assertion (3.17).

□

The following corollary gives relationships for the parametric Stieltjes, the parametric Laplace and the MacDonald function:

$$\mathrm{K}_\nu(x) = \sqrt{\pi}(2x)^\nu e^{-x}\,\mathrm{U}\left(\frac{1}{2}+\nu, 1+2\nu, 2x\right) \qquad \nu \geq 0, \tag{3.22}$$

(cf. [9, p. 528, (51:3:1)]).

Corollary 3.4. *We have*

$$\mathscr{S}_{\lambda_2,1}\big[\mathscr{L}_{\lambda_2}[g(y)](u)\big](z) = \Gamma(\lambda_2) z^{\lambda_2-1}\int_0^\infty y^{\lambda_2-1}\exp(zy)\;\Gamma(1-\lambda_2, zy) g(y)\,dy \tag{3.23}$$

$$\mathscr{S}_{\lambda_2,1-\lambda_2}\big[\mathscr{L}_{\lambda_2}[g(y)](u)\big](z) = \frac{\Gamma(\lambda_2)}{\pi^{1/2}} z^{\lambda_2-\frac{1}{2}}\int_0^\infty y^{-1/2}\exp\left(\frac{zy}{2}\right)\mathrm{K}_{\lambda_2-\frac{1}{2}}\left(\frac{zy}{2}\right) g(y)\,dy \tag{3.24}$$

Proof. In the relationship Equation (3.1) of Theorem 3.1 we set

$$f(x) = x^{1-\lambda_1}e^{-zx} \tag{3.25}$$

so that

$$\mathscr{L}_{\lambda_1}\big[f(x)\big](u) = (u+z)^{-1}, \tag{3.26}$$

and

$$\begin{aligned}\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y) &= \mathscr{S}_{\lambda_2}\big[x^{-\lambda_2}e^{-zx}\big](y)\\ &= \pi^{-1/2}\Gamma(1-\lambda_2)\left(\frac{z}{y}\right)^{\lambda_2-\frac{1}{2}}\exp\left(\frac{zy}{2}\right)\mathrm{K}_{\lambda_2-\frac{1}{2}}\left(\frac{zy}{2}\right),\end{aligned} \tag{3.27}$$

where we used the definition of the generalized Stieltjes transform Equation (1.5) and the result Equation [5, p. 233, 14.4 (9)]. Substituting the results Equations (3.28), (3.29), and (3.30) into the relationship Equation (3.1), we obtain the assertion Equation (3.23).

In the relationship Equation (3.1) of Theorem 3.1 we set

$$f(x) = x^{1-\lambda_1-\lambda_2}e^{-zx} \tag{3.28}$$

so that

$$\mathscr{L}_{\lambda_1}\big[f(x)\big](u) = \frac{\Gamma(1-\lambda_2)}{(u+z)^{1-\lambda_2}}, \tag{3.29}$$

and

$$\begin{aligned}\mathscr{S}_{\lambda_1,\lambda_2}\big[f(x)\big](y) &= \mathscr{S}_{\lambda_2}\big[x^{-\lambda_2}e^{-zx}\big](y)\\ &= \pi^{-1/2}\Gamma(1-\lambda_2)\left(\frac{z}{y}\right)^{\lambda_2-\frac{1}{2}}\exp\left(\frac{zy}{2}\right)\mathrm{K}_{\lambda_2-\frac{1}{2}}\left(\frac{zy}{2}\right),\end{aligned} \tag{3.30}$$

where we used the definition of the generalized Stieltjes transform Equation (1.5) and the result Equation [5, p. 233, 14.4 (9)]. Substituting the results Equations (3.28), (3.29), and (3.30) into the relationship Equation (3.1), we obtain the assertion Equation (3.23).

□

4. Some Illustrative Examples

An illustration of the relationship Equation (3.1) of Theorem 3.1 is to find an integration involving the Tricomi function.

Example 4.1. If $\Re(\lambda_1) > \Re(\lambda_1 - \lambda_2) > 0$, then we have

$$\int_0^\infty y^{\lambda_1-\mu-1}\mathrm{U}(\lambda_1, 1+\lambda_1-\lambda_2, \beta y)\,dy = \frac{\Gamma(\lambda_2-\mu)\mathrm{B}(\mu,\lambda_1-\mu)}{\Gamma(\lambda_2)}\beta^{\mu-\lambda_1}. \tag{4.1}$$

and

$$\int_0^\infty y^{z-1}e^y K_\nu(y)\,dy = \frac{\Gamma\left(\frac{1}{2}-z\right)\Gamma(z+\nu)\Gamma(z-\nu)}{\pi^{\frac{1}{2}}2^z\sec(\nu\pi)}. \tag{4.2}$$

Proof. In Equation (3.1) of Theorem 3.1 we set

$$f(x) = e^{-\beta x} \quad \text{and} \quad g(y) = y^{-\mu} \tag{4.3}$$

so that

$$\mathscr{L}_{\lambda_1}[f(x)](u) = \frac{\Gamma(\lambda_1)}{(u+\beta)^{\lambda_1}} \quad \text{and} \quad \mathscr{L}_{\lambda_2}[g(y)](u) = \frac{\Gamma(\lambda_2-\mu)}{u^{\lambda_2-\mu}}\cdot \tag{4.4}$$

Using the definition Equation (1.8) of the parametric Stieltjes transform, we have

$$\mathscr{S}_{\lambda_1,\lambda_2}[f(x)](y) = \int_0^\infty \frac{x^{\lambda_1-1}e^{-\beta x}}{(x+y)^{\lambda_2}} \tag{4.5}$$

Using the integral representation Equation (4.6) of the Tricomi function, we have

$$\mathscr{S}_{\lambda_1,\lambda_2}[f(x)](y) = \Gamma(\lambda_1)y^{\lambda_1-\lambda_2}\mathrm{U}(\lambda_1, 1+\lambda_1-\lambda_2, \beta y) \tag{4.6}$$

Substituting the results from Equations (4.3) and (4.5) into Equation (3.1) of Theorem 3.1 we obtain

$$\int_0^\infty y^{\lambda_1-\mu-1}\mathrm{U}(\lambda_1, 1+\lambda_1-\lambda_2, \beta y)\,dy = \frac{\Gamma(\lambda_2-\mu)}{\Gamma(\lambda_2)}\int_0^\infty \frac{u^{\mu-1}}{(u+\beta)^{\lambda_1}}\,du. \tag{4.7}$$

Now the assertion Equation (4.1) follows from the integral formula from [7, p.315, Entry 3.194 (3)] and Equation (4.7).

After $\lambda_1 = \nu + \frac{1}{2}$, $\lambda_2 = \frac{1}{2} - \nu$, $\mu = \frac{1}{2} - z$, and $\beta = 2$ in (4.1) of Example 4.1 and using the definition (3.22), Equation (4.1) becomes

$$\int_0^\infty y^{z-1}e^y\,\mathrm{K}_\nu(y)\,dy = \frac{\Gamma(z-\nu)\mathrm{B}\left(\frac{1}{2}-z, \nu+z\right)}{\pi^{\frac{1}{2}}2^z\Gamma\left(\frac{1}{2}-\nu\right)}\cdot \tag{4.8}$$

Now the assertion (4.2) follows from the definition of the Beta function (2.5) and the well-known property of the gamma function:

$$\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\nu\right) = \pi\sec(\pi\nu), \tag{4.9}$$

(cf. [9, Equation (43.5.2), p. 438]). □

Remark 4.1. Using the definition (1.11) of the Mellin transform, we can express Equation (4.2) as

$$\mathscr{M}\left[e^{y}K_{\nu}(y)\right](y)=\frac{\Gamma\left(\frac{1}{2}-z\right)\Gamma(z+\nu)\Gamma(z-\nu)}{\pi^{\frac{1}{2}}2^{z}\sec(\nu\pi)}, \tag{4.10}$$

(cf. [8, Entry (11.4), p. 115])

Example 4.2. If $\Re(\lambda_1)>\Re(\lambda_1-\lambda_2)>0$, then we have

$$\int_0^\infty y^{\lambda_1-1}e^{-\beta y}\mathrm{U}(\lambda_1,1+\lambda_1-\lambda_2,\alpha y)\,dy=\alpha^{-\lambda_1}B\left(\lambda_1,\lambda_2\right){}_2F_1\left(\lambda_1,\lambda_2;\lambda_1+\lambda_2;1-\frac{\beta}{\alpha}\right). \tag{4.11}$$

Proof. In Equation (3.1) of Theorem 3.1 we set

$$f(x)=e^{-\alpha x}\quad\text{and}\quad g(y)=e^{-\beta y} \tag{4.12}$$

Substituting the results from Equation (4.12) into Equation (3.1) of Theorem 3.1 we obtain

$$\int_0^\infty y^{\lambda_1-1}e^{-\beta y}\mathrm{U}(\lambda_1,1+\lambda_1-\lambda_2,\alpha y)\,dy=\Gamma(\lambda_1)\int_0^\infty\frac{u^{\lambda_2-1}}{(u+\alpha)^{\lambda_1}(u+\beta)^{\lambda_2}}\,du \tag{4.13}$$

Now the assertion Equation (4.11) follows from the integral formula from [7, p.317, Entry 3.197 (1)] and Equation (4.13). □

Example 4.3. If $\Re(\lambda_1)>\Re(\lambda_1-\lambda_2)>0$, then we have

$$\int_0^\infty y^{\lambda_2+\mu-1}e^{zy}\Gamma(1-\lambda_2,zy)\,dy=\frac{\Gamma(\lambda_2+\mu)}{\Gamma(\lambda_2)z^{\lambda_2+\mu}}B(-\mu,\mu+1). \tag{4.14}$$

Proof. In Equation (3.23) of Corollary (3.4) we set

$$g(y)=y^{\mu} \tag{4.15}$$

so that

$$\mathscr{L}_{\lambda_2}\left[g(y)\right](u)=\frac{\Gamma(\lambda_2+\mu)}{u^{\lambda_2+\mu}}. \tag{4.16}$$

Then

$$\mathscr{S}_{\lambda_2,1}\left[\mathscr{L}_{\lambda_2}\left[g(y)\right](u)\right](z)=\Gamma(\lambda_2+\mu)\int_0^\infty\frac{u^{-\mu-1}}{u+z}\,du. \tag{4.17}$$

Making the change of variable $t=u/z$ in the integral of Equation (4.17), we have

$$\mathscr{S}_{\lambda_2,1}\left[\mathscr{L}_{\lambda_2}\left[g(y)\right](u)\right](z)=\frac{\Gamma(\lambda_2+\mu)}{z^{\mu+1}}\int_0^\infty\frac{t^{-\mu-1}}{1+t}\,dt. \tag{4.18}$$

and using the definition of the Beta function (2.5), we have

$$\mathscr{S}_{\lambda_2,1}\left[\mathscr{L}_{\lambda_2}\left[g(y)\right](u)\right](z)=\frac{\Gamma(\lambda_2+\mu)}{z^{\mu+1}}B(-\mu,\mu+1). \tag{4.19}$$

Now the assertion (4.14) follows from Equation (4.19) and the relationship (3.23) of Corollary (3.4). □

Example 4.4. If $\Re(\lambda_1) > \Re(\lambda_1 - \lambda_2) > 0$, then we have

$$\int_0^\infty y^{\mu-\frac{1}{2}} \exp\left(\frac{zy}{2}\right) K_{\lambda_2-\frac{1}{2}}\left(\frac{zy}{2}\right) dy = \frac{\sqrt{\pi}\,\Gamma(\lambda_2+\mu)}{\Gamma(\lambda_2) z^{\mu+\frac{1}{2}}} B(-\mu, \mu-\lambda_2+1). \quad (4.20)$$

Proof. In Equation (3.24) of Corollary (3.4) we set

$$g(y) = y^\mu \quad (4.21)$$

so that

$$\mathscr{L}_{\lambda_2}\left[g(y)\right](u) = \frac{\Gamma(\lambda_2+\mu)}{u^{\lambda_2+\mu}}. \quad (4.22)$$

Then

$$\mathscr{S}_{\lambda_2,1-\lambda_2}\left[\mathscr{L}_{\lambda_2}\left[g(y)\right](u)\right](z) = \Gamma(\lambda_2+\mu) \int_0^\infty \frac{u^{-\mu-1}}{u+z}\, du. \quad (4.23)$$

Making the change of variable $t = u/z$ in the integral of Equation (4.23), we have

$$\mathscr{S}_{\lambda_2,1-\lambda_2}\left[\mathscr{L}_{\lambda_2}\left[g(y)\right](u)\right](z) = \frac{\Gamma(\lambda_2+\mu)}{z^{\mu-\lambda_2+1}} \int_0^\infty \frac{t^{-\mu-1}}{(1+t)^{1-\lambda_2}}\, dt. \quad (4.24)$$

and using the definition of the Beta function (2.5), we have

$$\mathscr{S}_{\lambda_2,1-\lambda_2}\left[\mathscr{L}_{\lambda_2}\left[g(y)\right](u)\right](z) = \frac{\Gamma(\lambda_2+\mu)}{z^{\mu-\lambda_2+1}} B(-\mu, \mu-\lambda_2+1). \quad (4.25)$$

Now the assertion (4.20) follows from Equation (4.25) and the relationship (3.23) of Corollary (3.4). □

5. Conclusion

The work being discussed introduces two important parametric integral transforms, namely the parametric Laplace transform and the parametric Stieltjes transform. These transforms are powerful mathematical tools used to study the behavior of functions and systems in various fields such as engineering, physics, and economics.

The work goes on to demonstrate how these integral transforms can be used to evaluate improper integrals of several special functions, including the MacDonald function, the Tricomi function, the exponential integral function, and the complementary error function. These special functions are widely used in many areas of mathematical physics, such as wave propagation, quantum mechanics, and statistical mechanics.

The approach taken in this work involves using Parseval-Goldstein type relations involving the parametric Laplace and parametric Stieltjes transforms. These relations allow for the transformation of a function from one domain to another, and are commonly used in mathematical analysis to simplify complex integrals and equations.

The key contribution of this work is that it demonstrates how these powerful mathematical tools can be used to evaluate integral transforms and improper integrals of well-known special functions in a relatively straightforward and elementary fashion. This has important implications for the study of complex systems and phenomena, as it provides researchers with new tools to analyze and understand these systems in greater depth.

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