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AUTOMORPHIC L-FUNCTIONS BY WAY OF THE KUZNETSOV SUM FORMULA

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Dedicated to Professor Dr. Takashi Ichikawa on his sixty-fifth birthday with great respect and friendship

Abstract We will give a smooth meromorphic continuation from holomorphic automorphic forms to non-holomorphic ones by computing the inner product of holomorphic Poincaré series with s modular form of weight k and that of two non-holomorphic Poincaré series by their Fourier expansion. Combining them via the Sears-Titchmarch expansion gives rise to the unfolding side of the inner product.
By the Parseval formula with respect to the spectral expansion we deduce the other side. Equating them gives Kuznetsov sum formula, i.e. an expression of the spectral sum with a general weight in terms of sums of Kloostermann sums.
Through careful analysis of the Kuznetsov sum formula we penetrate into the core of the theory with more ease and transparency.

Keywords Kuznetsov sum formula, zeta function, modular form, G-function

MSC(2010) 11S40, 11M41, 11F03

1. Overview

[39, p.1] states "A rather full but somewhat longwinded function-theoretic treatment of that case (the classical modular functions and modular forms) was given in 1890-1892 by R. Fricke and F. Klein; the arithmetical aspects, which are intimately tied up with the theory of complex multiplication, were considered by H. Weber in his Algebra, vol, III (for a modern treatment, cf. a forthcoming book by G. Shimura). **The relation between modular forms and Dirichlet series with functional equations was discovered by Hecke**, whose epoch-making work

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during the years 1930-1940,based on that discovery and that of the "Hecke operators," brought out completely new aspects of a theory which many mathematicians would have regarded as **a closed chapter long before**."

The relation is referred to as the **Riemann-Hecke-Bochner correspondence** which has been developed by Bochner, Chandrasekharan and Narasimhan, M. Knopp, B. Berndt et al and has been culminated in [15].

The spectral theory of automorphic forms may be said to begin with the discovery of Maass (wave) forms [20]. In pursuit of extending Hecke's result on the L-functions associated to *holomorphic* automorphic forms, Maass went backwards from the Dirichlet series with Grössen-characters attached to real quadratic fields and introduced *non-holomorphic* automorphic forms, eigenfunctions of the hyperbolic Lalacian Δ, the Maass (wave) forms.

Selberg [32] and Roelcke [29] independently developed the L^2-theory of automorphic forms and established spectral expansions. Combined with the development of representation theory notably by [11], the theory has been vastly advanced.

In [1, p.2], R. Baker mentions "The idea is to make a difficult subject a bit easier for beginners." There are a few books [5], [12], [24], [37], [38] devoted to the subject but none of them are easy to read. In these lectures, we concentrate on the Kuznetsov sum formula as opposed to the Kuznetsov trace formula [17] and pave a light-hearted promenade to understanding of the subject to such an extent that one can go on reading more advanced material and possibly come to the front to make research. Selberg introduced [33] a very important class of series, known as Poincaré series, which led him to introduce a new class of zeta-functions, the Kloostermann(-Selberg) zeta-functions. Kuznetsov-Motohashi established the trace formula by equating two expressions for the inner product of two Poincaré series, one by the unfolding method and the other by the Parseval formula. The unfolding method is ubiquitous starting probably from Petersson's use [25] in computation of the Fourier coefficients of a modular form of weight k.

2. Introduction

In pursuit of elucidation of the Kuznetsov sum formula [22], it has turned out that the key idea due to Selberg [33] ([32]) of expressing the inner product of two Poincaré series in two ways has a catalytic effect in the modified argument of Motohashi.

$$(P_m(\cdot, s_1), P_n(\cdot, \bar{s}_2)) = \int_{\mathcal{F}} P_m(z, s_1)\overline{P_n(z, \bar{s}_2)}\,\mathrm{d}z,$$

where $P_m(z, s)$ is the real analytic Poincaré series defined on [38, p.36] which is referred to in [14]. We use (6.3) below which is a slightly generalized one than that from [24, p.4].

The inner product of Poincaré series is computed in Motohashi [24, p.44] for integers $m, n > 0$. Motohashi's method amounts to the elimination of the

$G^{2,0}_{1,3}\left(\frac{4\pi^2 mn}{c^2} \,\middle|\, \begin{matrix} s_1 \\ 0, s_2 - s_1, 1 - s_1 \end{matrix}\right)$-term from the Neumann series part in Theorem 7.1 and expressing the Kloostermann sum zeta-function in terms of the inner product of two Poincaré series. Then substituting the spectral result for the same arising from the Parseval formula. thus expressing the Kloostermann zeta-function in terms of spectra, discrete, continuous and holomorphic.

Equating the Unfolding formula and the Parseval formula, we may deduce the Kuznetsov sum formula in an accessible way. For the proof cf. [18] and [14]. During elucidation of the Kuznetsov sum formula, it has become clear that there are many instances in the theory of automorphic L-functions where known results are not mentioned but proved by another method. One typical example is the case of the Chowla-Selberg (integral) formula [7], [34].

The **holomorphic Eisenstein series** is defined by

$$\zeta_{\mathbb{Z}^2}(2s;\ \alpha) = \sum_{m,n=-\infty}^{\infty}{}' |m + n\alpha|^{-2s} \tag{2.1}$$

where $\operatorname{Re} s = \sigma > 1$, $\alpha = x + iy$ $(y > 0)$ and the prime on the summation sign means the omission of the term with $m = n = 0$. Since the summand is $\left|(m + nx)^2 + n^2y^2\right|^{-1} = Q(m, n)$, with $Q(m, n) = (x^2 + y^2)n^2 + 2xmn + m^2$, the holomorphic Eisenstein series is an **Epstein zeta-function associated with a positive definite binary quadratic form** defined by (2.1):

$$\zeta_{\mathbb{Z}^2}(2s;\ \alpha) = Z(s, Q), \quad Q(m, n) = (x^2 + y^2)n^2 + 2xmn + m^2.$$

Zhang and Williams [40] *reduce the Epstein zeta-function to a holomorphic Eisenstein series* and apply the Poisson summation formula to deduce the Chowla-Selberg integral formula.

As can be seen from (2.1), there are some cases where the non-holomorphic automorphic forms are *constructed by just multiplying the non-holomorphic factor* y^s:

$$E(z, s) = \frac{y^s}{2\zeta(2s)}\zeta_{\mathbb{Z}^2}(2s;\ z). \tag{2.2}$$

We note that the factor $\frac{1}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(\ell)}{\ell^{2s}}$ has the *effect of restricting the summation variables to those which are relatively prime.* This is one form of the **relative primality principle** [19, p.75]. Indeed, multiplying (2.1) by this, we obtain

$$\begin{aligned} \frac{1}{\zeta(2s)}\zeta_{\mathbb{Z}^2}(2s;\ z) &= \sum_{\ell,m,n=-\infty}^{\infty}{}' |\ell m + \ell nz|^{-2s} \\ &= \sum_{m,n=-\infty}^{\infty}{}' \sum_{\ell|(m,n)} \mu(\ell)|m + nz|^{-2s} \\ &= \sum_{\substack{m,n=-\infty \\ (m,n)=1}}^{\infty}{}' |m + nz|^{-2s}. \end{aligned} \tag{2.3}$$

This leads to (2.2), i.e.

$$E(z, s) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im}\gamma z)^s = \frac{y^s}{2\zeta(2s)}\zeta_{\mathbb{Z}^2}(2s;\ z),$$

i.e. *the theory of* $E(z,s)$ *is that of* $\frac{1}{2\zeta(2s)}\zeta_{\mathbb{Z}^2}(2s;\ z)$ *multiplied by* y^s. It may therefore reduce to this theory if we restrict to the full modular group. To avoid this, we consider the case where there are more cusps than one, denoting them by German alphabets.(*cf.* Theorem 7.1)

As another example we take up the Poincaré series holomorphic and non-holomorphic, (4.5). It seems that there are some hidden facts here and our objective is to find out such and also an object corresponding to the functional equation.

Once the functional equation is known, then all these are unified in the framework of "The Fourier-Bessel expansion $G^{1,1}_{1,1} \leftrightarrow G^{2,0}_{0,2}$" [15, Chapter 4].

name	holom.	non-holom.
Eisenstein ser.	$\zeta_{\mathbb{Z}^2}(2s;\alpha) = Z(s,Q_1)$	$E(z,s) = \frac{y^s}{2\zeta(2s)}\zeta_{\mathbb{Z}^2}(2s;z)$
Epstein zeta	$Z(s,Q) = \sum'_{m,n} \frac{1}{Q(m,n)^s}$	$\frac{y^s}{2\zeta(2s)} Z(s,Q_1)$
Poincaré ser.	$U_{\mathfrak{a}m}(z)$, (2.7)	$E_{\mathfrak{a}m}(z\vert\psi)$

Table 1. Holomorphic, non-holomorphic Maass forms

2.1. Modular forms: a titbit

Although we mainly treat discrete subgroups of

$$SL(2,\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R}, ad-bc = 1 \right\},$$

expecting future generalization, we state some results of discrete subgroups of $G = PGL(2,\mathbb{R})$. (S and G are for special and general). We may let $SL(2,\mathbb{R})$ act on $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the linear fractional transformation (or Möbius transformation):

$$\gamma z = \frac{az+b}{cz+d} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \in \mathcal{H}.$$

It can be immediately checked that the imaginary part $y(\gamma z)$ of γz satisfies

$$y(\gamma z) = \operatorname{Im}\gamma(z) = \frac{y}{|cz+d|^2} = \frac{y}{|j_\gamma(z)|^2} = \frac{y}{|cz+d|^2},$$

so that the upper half-plane $\mathcal{H}$ (or the Lobačevskiĭ) is stable under the action of $SL(2,\mathbb{R})$.

Remark 2.1. In what follows we often define a function involving the complex power of $y(\gamma z)$, which are of course not holomorphic in z.

Let Γ denote the Fuchsian group of the first kind (a discrete subgroup of $G = PGL(2,\mathbb{R})$ with $\Gamma\backslash G$ non-compact and its fundamental domain of finite volume) and let $\mathcal{M}_k(\Gamma)$ (M is for modular) be the linear space of all *holomorphic* functions $f : \mathcal{H} \to \mathbb{C}$ satisfying the automorphic property

$$j_\gamma(z)^{-k} f(\gamma z) = f(z), \qquad \gamma \in \Gamma, \tag{2.4}$$

where j is the denominator:

$$j_\gamma(z) = cz + d,$$

with a convention that $c \geq 0$ and if $c = 0$, then $d > 1$. More concretely (2.4) reads

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z), \quad z \in \mathcal{H}, \quad \gamma \in \Gamma.$$

The element f of $\mathcal{M}_k(\Gamma)$ is called an **automorphic form** (or a modular form) of weight k with respect to Γ, where $k \geq 0$ is an *even* integer (or some write $2k$ for k).

One motivation for considering condition (2.4) can be found in [35, p.80].

We have

$$\frac{\mathrm{d}(\gamma z)}{\mathrm{d}z} = \frac{1}{j_\gamma(z)^2}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and so (2.4) reads

$$\frac{f(\gamma z)}{f(z)} = \left(\frac{\mathrm{d}(\gamma z)}{\mathrm{d}z}\right)^{-\frac{k}{2}}$$

or

$$f(\gamma z)\,\mathrm{d}(\gamma z)^{\frac{k}{2}} = f(z)\,\mathrm{d}z^{\frac{k}{2}},$$

i.e. the differential form $f(z)\,\mathrm{d}z^{\frac{k}{2}}$ of degree $\frac{k}{2}$ is invarianat under Γ.

In the case of the full modular group $\Gamma = PSL_2(\mathbb{Z})$, it is generated by the translation and the Spiegelung

$$Tz = z + 1, \quad Sz = -\frac{1}{z}$$

and the translation generates the stabilizer of ∞.

$$\Gamma_\infty = \langle T \rangle = \{T^n \mid n \in \mathbb{Z}\}. \tag{2.5}$$

For each cusp $\mathfrak{a}$ we fix the element $\sigma_\mathfrak{a} \in SL_2(\mathbb{R})$ such that

$$\sigma_\mathfrak{a}\infty = \mathfrak{a}, \quad \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty,$$

where $\Gamma_\mathfrak{a}$ is the stabilizer of $\mathfrak{a}$ (parabolic subgroup)

$$\Gamma_\mathfrak{a} = \{\gamma \in \Gamma \mid \gamma\mathfrak{a} = \mathfrak{a}\}.$$

Every $f \in \mathcal{M}_k$ has the **Fourier expansion**

$$j_{\sigma_\mathfrak{a}}(z)^{-k} f(\sigma_\mathfrak{a} z) = \sum_{n=0}^{\infty} \hat{f}_\mathfrak{a}(n) e(nz), \quad e(z) = e^{2\pi i z} \tag{2.6}$$

which converges absolutely and uniformly on compact subsets. If at every cusp $\mathfrak{a}$, the constant term vanishes

$$\hat{f}_\mathfrak{a}(0) = 0,$$

then f is called a **cusp form** (or a Spitzenform). A cusp form is of exponential decay at cusps, and in particular $y^{k/2} f(z)$ is bounded on $\mathcal{H}$. Let $\mathcal{S}_k(\Gamma)$ (S is for

Spitzenform) denote the space of all cusp forms of weight k which is equipped with the **Petersson inner product** (1939)

$$\langle f, g\rangle_k = \int_{\mathcal{F}} y^k f(z)\overline{g(z)}\,\mathrm{d}z$$

which makes sense in $\mathcal{M}_k(\Gamma)$ for $k > 2$.

Theorem 2.1. *$\mathcal{S}_k(\Gamma)$ is spanned by holomorphic Poincaré series (in contrast to non-holomorphic ones in (4.4)):*

$$U_{\mathfrak{a}m}(z,k) = \sum_{\gamma\in\Gamma_\alpha\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma z). \tag{2.7}$$

Proof. The subspace spanned by the holomorphic Poincaré series is closed and any function orthogonal to the subspace is 0 by Petersson's formula (3.1) below. □

In the case $\mathfrak{a} = \infty$, (2.7) reduces to

$$U_m(z,k) = U_{\infty m}(z,k) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_\gamma(z)^{-k} e(m\gamma(z)).$$

3. Petersson trace formula→ Neumann series

The unfolding method of Rankin [28] and Selberg [31] has been extensively used in spectral theory of automorphic functions.

As mentioned above, Petersson [25] already used unfolding method to prove the formula for the Fourier coefficients in (2.6).

$$\hat{f}_{\mathfrak{a}}(n) = \frac{(4\pi n)^{k-1}}{(k-2)!}\langle f, U_{\mathfrak{a}n}(\cdot,k)\rangle_k, \tag{3.1}$$

and obtained

$$j_{\sigma_{\mathfrak{b}}}(z)^{-k} U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z,k) = \sum_{n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \hat{U}_{\mathfrak{a}\mathfrak{b}}(m,n) e(nz) \tag{3.2}$$

with

$$\hat{U}_{\mathfrak{a}\mathfrak{b}}(m,n) = \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn} + 2\pi i^k \sum_c \frac{S_{\mathfrak{a}\mathfrak{b}}(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{3.3}$$

Let $\{f_{jk}\}_{1\le j\le \vartheta_k}$ be an orthonormal basis of $\mathcal{M}_k(\Gamma)$ and let

$$f_{jk}(z) = \sum_{n=1}^{\infty} \hat{f}_{\mathfrak{a}jk}(n) e(nz)$$

be the expansion at the cusp $\mathfrak{a}$. Let

$$U_{\mathfrak{a}m}(z,k) = \frac{(k-2)!}{(4\pi m)^{k-1}} \sum_j \overline{\hat{f}_{\mathfrak{a}jk}(m)} f_{jk}(z)$$

be the expansion of Poincaré series with respect to this basis. We write this as

$$\begin{aligned} U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}z,k) &= \frac{(k-2)!}{(4\pi m)^{k-1}}\sum_j \overline{\hat{f}_{\mathfrak{a}jk}}(m) f_{jk}(\sigma_{\mathfrak{b}}z) \\ &= \frac{(k-2)!}{(4\pi m)^{k-1}}\sum_j \overline{\hat{f}_{\mathfrak{a}jk}}(m)\sum_{n=1}^{\infty}\hat{f}_{\mathfrak{b}jk}(n)e(nz)j_{\sigma_{\mathfrak{b}}}(z)^k. \qquad (3.4)\end{aligned}$$

Equating (3.2) and (3.4) multiplied by $j_{\sigma_{\mathfrak{b}}}(z)^{-k}$, we conclude that

$$\begin{aligned} &\frac{(k-2)!}{(4\pi m)^{k-1}}\sum_{n=1}^{\infty}\sum_j \overline{\hat{f}_{\mathfrak{a}jk}}(m)\hat{f}_{\mathfrak{b}jk}(n)e(nz) \\ &= j_{\sigma_{\mathfrak{b}}}(z)^{-k}U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}z,k) = \sum_{n=1}^{\infty}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\hat{U}_{\mathfrak{a}\mathfrak{b}}(m,n)e(nz),\end{aligned}$$

whence that

$$\frac{(k-2)!}{(4\pi m)^{k-1}}\sum_j \overline{\hat{f}_{\mathfrak{a}jk}}(m)\hat{f}_{\mathfrak{b}jk}(n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}}\hat{U}_{\mathfrak{a}\mathfrak{b}}(m,n)$$

or

$$\frac{(k-2)!}{(4\pi\sqrt{mn})^{k-1}}\sum_j \overline{\hat{f}_{\mathfrak{a}jk}}(m)\hat{f}_{\mathfrak{b}jk}(n) = \hat{U}_{\mathfrak{a}\mathfrak{b}}(m,n). \qquad (3.5)$$

Equating (3.5) and (3.3), we deduce **Petersson's trace formula**

Theorem 3.1. *Let m, n be positive integers and k a positive even integer. Then*

$$\begin{aligned} &\sum_c \frac{S_{\mathfrak{a}\mathfrak{b}}(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ &= \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}\frac{(-1)^{\frac{k}{2}-1}}{2\pi} + \frac{(-1)^{\frac{k}{2}}}{2\pi}\frac{(k-2)!}{(4\pi\sqrt{mn})^{k-1}}\sum_j \overline{\hat{f}_{\mathfrak{a}jk}}(m)\hat{f}_{\mathfrak{b}jk}(n) \qquad (3.6)\end{aligned}$$

The name is stated only on [24, p.51], in no other books have this terminology. Some papers refer to the Petersson trace formula and refer to [13, Theorem 3.6, p.54] in which there is no mention of this terminology, but stated as Petersson's formulas.

Let $f(x)$ be a continuous function of bounded variation on $\mathbb{R}_+$ such that

$$\int_0^{\infty}|f(x)|x^{-1/2}\,\mathrm{d}x < \infty$$

in particular $f(x)$ may be an infinitely many times differentiable function with compact support. We follow [12] which gives the clearest exposition thereof.

Let f^0 be the projection of f on the space spanned by odd indexed Bessel functions $\{J_{2n+1}|n \geq 0\}$ and is given by the *Neumann series*

$$f^0(x) = \sum_{n=0}^{\infty}2(2n+1)J_{2n+1}(x)N_f(2n+1)$$

and

$$N_f(\lambda) = \int_0^\infty J_\lambda(y) f(y) \,\frac{\mathrm{d}y}{y}$$

is the *Neumann integral*. In [37, p.36] this formula is stated with a typo of $2ir$ which should be $2n+1$.

Multiplying (3.6) by $2(2k-1)N_f(2k-1)$ and summing over $k = 1, 2, \cdots$, we conclude

Theorem 3.2. (Iwaniec [12, Theorem 9.6]) *Let $\mathfrak{a}$, $\mathfrak{b}$ be cusps of the Fuchsian group of the first kind Γ and let $m, n > 0$ be integers. Then for any test function f satisfying the condition*

$$f(0) = 0, \quad f^{(j)}(x) \ll (x+1)^{-j-1}, \quad j = 0, 1, 2, \tag{3.7}$$

we have

$$\begin{aligned} &- \delta_{\mathfrak{a}\mathfrak{b}} \delta_{mn} f^\infty + \sum_c c^{-1} \mathcal{S}_{\mathfrak{a}\mathfrak{b}}(m, n; c) f^0 \left(\frac{4\pi\sqrt{|mn|}}{c} \right) \\ &= \sum_{k=1}^\infty i^{2k} N_f(2k-1) \bar{\psi}_{\mathfrak{a}j2k}(m) \psi_{\mathfrak{b}j2k}(n), \end{aligned}$$

where $f^\infty = f - f^0$ and $\psi_{\mathfrak{a}jk(m)}$ are the normalized Fourier coefficients

$$\psi_{\mathfrak{a}jk}(m) = \left(\frac{\pi^{-k}\Gamma(k)}{(4m)^{k-1}} \right)^{1/2} \hat{f}_{\mathfrak{a}jk}(m).$$

This is the exact complement to Theorem 4.2.

Proof. Proof of (3.1) by unfolding method. We restrict to the case $\mathfrak{a} = \infty$. Then

$$\langle f, U_m(\cdot, k) \rangle_k = \int_{\Gamma\backslash\mathcal{H}} \sum_{m=0}^\infty \hat{f}(n) e^{2\pi i m z} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\operatorname{Im}\gamma z} \cdot \overline{e^{2\pi i n z}} y^{k-1} \,\mathrm{d}x\mathrm{d}y.$$

Since $\int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} = \int_{\Gamma_\infty\backslash\mathcal{H}}$ (unfolding method) and the series in the integrand are of exponential deacy in y, we may freely interchange the summation and integration to deduce

$$\begin{aligned} \langle f, U_m(\cdot, k) \rangle_k &= \sum_{n=0}^\infty \hat{f}(n) \int_{\Gamma_\infty\backslash\mathcal{H}} e^{2\pi i(m-n)x} \,\mathrm{d}x y^{\bar{s}+k-2} e^{2\pi(m+n)y} \,\mathrm{d}y \\ &= \sum_{n=0}^\infty \delta_{mn} \int_0^\infty y^{\bar{s}+k-2} e^{2\pi(m+n)y} \,\mathrm{d}y \end{aligned}$$

by first integrating $\int_0^1 e^{2\pi i(m-n)x} \,\mathrm{d}x$. Hence the only $n = m$ term survives to give

$$\hat{f}(m) \int_0^\infty y^{\bar{s}+k-2} e^{2\pi(m+n)y} \,\mathrm{d}y = \frac{\hat{f}(m)}{4\pi m^{\bar{s}+k-1}} \Gamma(\bar{s}+k-1)$$

for $\operatorname{Re}\bar{s} + k - 1 > 0$ whence in particular (3.1). □

Proof. Proof of Petersson's results. We prove (3.2) and (3.3) by the double coset decomposition and Poisson summation formula. Recalling

$$j(\sigma_{\mathfrak{b}}(z))^{-k}U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}z,k) = \sum_{\gamma\in\Gamma_\alpha\backslash\Gamma} j(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)^{-k}e(m\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)$$
$$= \sum_{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}} g(\gamma z), \qquad (3.8)$$

where

$$g_0(z) = j(z)e^{2\pi imz}, \quad g(\gamma z) = g_0(\gamma z),$$

and where $j(\gamma z) = j_\gamma(z)$.

We have [1, p.121]

$$\gamma_{d/c}T^n z = \frac{d^*}{c} - \frac{1}{c^2}\frac{1}{n+x+\frac{d}{c}+iy},$$

where d^* is in (6.1):

$$\gamma_{d/c} = \begin{pmatrix} d^* & b \\ c & d\end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}},$$

so that

$$g(\gamma_{d/c}T^n z) = c^{-k}\left(n+x+\frac{d}{c}+iy\right)^{-k} e^{2\pi i\frac{d^*m}{c}} e^{2\pi i\frac{-mc^{-2}}{n+x+\frac{d}{c}+iy}}.$$

Hence

$$\sum_{n=-\infty}^{\infty} g(\gamma_{d/c}T^n z) = e^{2\pi i\frac{d^*m}{c}} \sum_{n=-\infty}^{\infty} c^{-k}\left(n+x+\frac{d}{c}+iy\right)^{-k} e^{2\pi i\frac{-mc^{-2}}{n+x+\frac{d}{c}+iy}}$$
$$= e^{2\pi i\frac{d^*m}{c}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} c^{-k}\left(t+x+\frac{d}{c}+iy\right)^{-k} e^{2\pi i\frac{-mc^{-2}}{t+x+\frac{d}{c}+iy}-2\pi int}\,\mathrm{d}t$$

by the Poisson summation formula. on putting $t+x+\frac{d}{c} = u$, the integral becomes

$$e^{2\pi i\frac{dn}{c}} e^{2\pi inx} I(m,n,c,y),$$

where

$$I(m,n,c,y) = \int_{-\infty}^{\infty} c^{-k}(u+iy)^{-k} e^{2\pi i\frac{-mc^{-2}}{u+iy}-2\pi inu}\,\mathrm{d}u. \qquad (3.9)$$

or

$$e^{2\pi i\frac{dn}{c}} e^{2\pi inz} I_1(m,n,c,y),$$

where

$$I_1(m,n,c,y) = \int_{-\infty}^{\infty} (u+iy)^{-k} e^{2\pi i\frac{-mc^{-2}}{u+iy}-2\pi in(u+iy)}\,\mathrm{d}u. \qquad (3.10)$$

Hence the right-hand side of (6.2) becomes

$$\delta_{\mathfrak{a}\mathfrak{b}}e^{2\pi imz} + \sum_{c=1}^{\infty} c^{-k} \sum_{n=-\infty}^{\infty} \sum_{d(\mathrm{mod} c)} e^{2\pi i\frac{d^*m+dn}{c}} I_1(m,n,c,y)e^{2\pi inz},$$

Noting that the definition (5.8) of Kloostermann sum amounts to

$$S_{\mathfrak{ab}}(m,n;c) = \sum_{d(\mathrm{mod} c)} e^{2\pi i \frac{md^*+nd}{c}},$$

(3.8) can be expressed as

$$j(\sigma_{\mathfrak{b}}(z))^{-k} U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z, k) = \delta_{\mathfrak{ab}} e^{2\pi i m z} + \sum_{c=1}^{\infty} c^{-k} \sum_{n=-\infty}^{\infty} S_{\mathfrak{ab}}(m,n;c) I_1(m,n,c,y) e^{2\pi i n z}. \tag{3.11}$$

Recall the integral representation

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{\nu} \int_{c-i\infty}^{c+i\infty} e^{t-\frac{z^2}{4t}} t^{-\nu-1}\, \mathrm{d}t$$

for $\mathrm{Re}\,\nu > 1$ and $c > 0$.

Incorporating the change of variable $t = -2\pi i n(u+iy)$ in (3.10), we find that

$$I_1(m,n,c,y) = 2\pi(-i)^k \left(c\sqrt{\frac{n}{m}}\right)^{k-1} \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{k-1} \int_{2\pi n y - i\infty}^{2\pi n y + i\infty} t^{-(k-1)-1} e^{t-\frac{z^2}{4t}}\, \mathrm{d}t,$$

where $z = \frac{4\pi\sqrt{mn}}{c}$. Hence

$$I_1(m,n,c,y) = 2\pi(-i)^k \left(c\sqrt{\frac{n}{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{3.12}$$

Substituting (3.12) in (3.11), we deduce that

$$\begin{aligned} & j(\sigma_{\mathfrak{b}}(z))^{-k} U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z, k) = \delta_{\mathfrak{ab}} e^{2\pi i m z} \\ & + 2\pi(-i)^k \sum_{c=1}^{\infty} c^{-k} \sum_{n=-\infty}^{\infty} \left(c\sqrt{\frac{n}{m}}\right)^{k-1} S_{\mathfrak{ab}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i n z}. \end{aligned}$$

We may put this in the form (3.2) with $\tilde{U}_{\mathfrak{ab}}$ as in (3.3), completing the proof. □

4. Kuznetsov trace formula

Definition 4.1. If a function $f : \mathcal{H} \to \mathbb{C}$ satisfies the condition (2.4) with $k = 0$, then it is called a **Γ-automorphic function** (or automorphic):

$$f(\gamma z) = f(z), \qquad \gamma \in \Gamma.$$

The space of all automorphic functions is denoted $\mathcal{A}(\Gamma\backslash\mathcal{H})$. An $f \in \mathcal{A}(\Gamma\backslash\mathcal{H})$ is an **automorphic form** (of Maass) if it is an eigenfunction of the non-Euclidean Laplacian:

$$(\Delta + \lambda) f = 0, \quad \lambda = s(1-s).$$

Some authors use the minus sign. If an automorphic form has of polynomial growth at each cusp of $\Gamma\backslash\mathcal{H}$, it is called a **Maass form**.

A Poincaré series (or incomplete theta series) at the cusp $\mathfrak{a}$ is defined by changing ∞ in (5.5) by $\mathfrak{a}$:

$$E_{\mathfrak{a}}(z|f) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} f(\sigma_{\mathfrak{a}}^{-1} \gamma z),$$

where f is periodic $f(Tz) = f(z)$ and subject to suitable growth condition. We write

$$E(z|f) = E_{\infty}(z|f) = P_f(z).$$

Corresponding to (5.4), one often considers the case

$$f(z) = \psi(y) e^{2\pi i m z}, \tag{4.1}$$

where m is a non-negative integer and $\psi : \mathbb{R}_+ \to \mathbb{C}$ satisfies the condition for some $a > 1$

$$\psi(y) \ll y |\log y|^{-a} \quad (0 < y < 1).$$

Suppose Γ has cusps. At a cusp $\mathfrak{a}$ any automorphic function f satisfies

$$f(\sigma_{\mathfrak{a}} T^n z) = f(\sigma_{\mathfrak{a}} z).$$

Hence it has a Fourier expansion

$$f(\sigma_{\mathfrak{a}} z) = \sum_{n=-\infty}^{\infty} f_{\mathfrak{a}n}(y) e^{2\pi i n x}, \tag{4.2}$$

where the nth coefficient is given by

$$f_{\mathfrak{a}n}(y) = \int_0^1 f(\sigma_{\mathfrak{a}} z) e^{-2\pi i n x} \, \mathrm{d}x. \tag{4.3}$$

If f is smooth, (4.2) converges absolutely and uniformly on compact sets. The 0th Fourier coefficient $f_{\sigma_{\mathfrak{a}}} = f_{\sigma_{\mathfrak{a}} 0}$ identically vanishes, then f is called a **cusp form**.

On [38, p.36] the *real analytic Poincaré series with character* is defined

$$P_m(z, s, \chi) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \bar{\chi}(\gamma) y(\gamma z)^s e^{2\pi i (m - \xi) \gamma z}, \quad z \in \mathcal{H}, \tag{4.4}$$

where χ is a one-dimensional unitary representation of Γ. Here $\xi \in \mathbb{R}$ is defined in the setting of (2.1) as follows.

$$\chi({\sigma_{\mathfrak{a}}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}}) = e^{2\pi i \xi}.$$

$P_m(z, s, 1)$ (i.e. $\xi = 0$) reduces to the one defined on [24, p.4]

$$P_m(z, s) = P_m(z, s, 1) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} y(\gamma z)^s e^{2\pi i m \gamma(z)}, \quad z \in \mathcal{H}.$$

Only on the real axis, there is an equality

$$y^{2k} U_m(z, k) = P_m(z, 2k), \quad z \in \mathbb{R}. \tag{4.5}$$

Now we consider the general Poincaré series with (4.1) incorporated:

$$E_{\mathfrak{a}m}(z|\psi) = E_{\mathfrak{a}m}(z|f) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\operatorname{Im}\sigma_{\mathfrak{a}}^{-1}\gamma z)e^{2\pi i m(\sigma_{\mathfrak{a}}^{-1}\gamma z)} \tag{4.6}$$

so that $E_{\infty m}(z|\cdot^s) = P_m(z,s)$. We write

$$E_{\mathfrak{a}}(z|\psi) = E_{\mathfrak{a}0}(z|\psi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\operatorname{Im}\sigma_{\mathfrak{a}}^{-1}\gamma z)$$

Proposition 4.1. (i) *$E_{\mathfrak{a}m}(z|\psi)$ is absolutely convergent on $\mathcal{H}$.*
(ii) *ψ is of compact support, then $E_{\mathfrak{a}m}(z|\psi)$ is bounded on $\mathcal{H}$.*

Two important special cases with $m = 0$ are:

$$E_{\mathfrak{a}}(z,s) = E_{\mathfrak{a}}(z|\cdot^s) = E_{\mathfrak{a}0}(z|\cdot^s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} (\operatorname{Im}\sigma_{\mathfrak{a}}^{-1}\gamma z)^s$$

–the Eisenstein series.
If $f \in C^\infty(\mathbb{R}_+)$ is compactly supported, then $E_{\mathfrak{a}m}(z|\psi)$ is called an **incomplete theta series** and the linear space of all incomplete theta series is denoted $\mathcal{E}(\Gamma\backslash\mathcal{H})$.

choice	f	$f = \psi\cdot e$	$\psi(y) = y^s$	$m = 0$
$\mathfrak{a}$	$E_{\mathfrak{a}}(z\|f)$	$P_{\mathfrak{a}m}(z\|\psi)$	$E_{\mathfrak{a}m}(z,\cdot^s)$	$E_{\mathfrak{a}}(z,s)$
$\mathfrak{a}=\infty$	$E(z\|s) \approxeq P_f(z)$	$P_f(g)$	$P_m(z,s)$	$E(z,s)$

Table 3. Non-holomorphic Poincaré (Eisenstein) series

As described on [12, pp.141-147] and on [1, pp.16-18], the reverse Kuznetsov trace formula is to be regarded as an expansion in J-Bessel functions due to Sears and Titchmarsh [30, (4.4)] and in many literature this reversed form is referred to as the Kuznetsov summation formula, [1], [21], [37]. From representation-theoretic point of view, this is natural since this expresses the weighted Kloostermann sum in terms of spectral part consisting of discrete and continuous spectra plus Neumann series part arising from the holomorphic modular forms.

The proof of the Kuznetsov sum formula in the Neumann series part has been done by Sears and Titchmarsh and Kuznetsov rediscovered it. Kuznetsov's contribution lies in the combination of the Neumann series part with the spectral part which corresponds to f^∞ under Sears-Titchmarsh inversion.

To state the spectral part, let

$$B_\nu(x) = \frac{1}{2\sin\frac{\pi}{2}\nu}\left(J_{-\nu}(x) - J_\nu(x)\right)$$

and define the *Titchmarsh integral* $T_f(t)$ by

$$T_f(t) = \int_0^\infty f(x)B_{2it}(x)\,\frac{\mathrm{d}x}{x} = \int_0^\infty f(x)\frac{J_{-2it}(x) - J_{2it}(x)}{2\sinh\pi t}\,\frac{\mathrm{d}x}{x}.$$

Then define the continuous superposition of projections of f on B_{2it} by

$$f^{\infty}(x) = \int_0^{\infty} T_f(t) B_{2it}(x) \tanh(\pi t)\,\mathrm{d}t = \int_0^{\infty} T_f(t) \frac{J_{-2it}(x) - J_{2it}(x)}{2\cosh(\pi t)}\,\mathrm{d}t.$$

Theorem 4.1. (Sears-Titchmarsh inversion) *We have the Sears-Titchmarsh inversion*

$$f = f^0 + f^{\infty}.$$

Also define the constant

$$f^{\infty} = \frac{1}{2\pi} \int_0^{\infty} T_f(t) \tanh(\pi t)\,\mathrm{d}t.$$

Theorem 4.2. (Iwaniec [12, Theorem 9.5]) *Let $\mathfrak{a}$, $\mathfrak{b}$ be cusps of the Fuchsian group of the first kind Γ and let $m, n > 0$ be integers. Then for any test function f satisfying the condition* (3.7)*, we have*

$$\begin{aligned}&\delta_{\mathfrak{a}\mathfrak{b}} \delta_{mn} f^{\infty} + \sum_c c^{-1} \mathcal{S}_{\mathfrak{a}\mathfrak{b}}(m, n; c) f^{\infty}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \\ &= \sum_j T_f(t_j) \bar{\nu}_{\mathfrak{a}}(m) \nu_{\mathfrak{b}j}(n) + \sum_{\mathfrak{c}} \frac{1}{4\pi} \int_{-\infty}^{\infty} T_f(t) \bar{\eta}_{\mathfrak{a}j\mathfrak{c}}(m, t) \eta_{\mathfrak{b}\mathfrak{c}}(n, t)\,\mathrm{d}t\end{aligned}$$

It seems that the corresponding formulas in [1] are incorrect in comparison with other refs.

Adding formulas in Theorems 4.2 and 3.2 in the light of the Sears-Titchmarsh inversion gives the reversed Kuznetsov sum formula in contrast to Theorem 4.3.

Theorem 4.3. (Iwaniec [12, Theorem 9.5]) *Let $\mathfrak{a}$, $\mathfrak{b}$ be cusps of the Fuchsian group Γ of the first kind and let $m, n > 0$ be integers. Then for any test function h satisfying the condition* (3.7)*, we have*

$$\begin{aligned}&\sum_c c^{-1} \mathcal{S}_{\mathfrak{a}\mathfrak{b}}(m, n; c) f\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \\ &= \sum_j T_f(t_j) \bar{\nu}_{\mathfrak{a}j}(m) \nu_{\mathfrak{b}j}(n) + \sum_{\mathfrak{c}} \frac{1}{4\pi} \int_{-\infty}^{\infty} T_f(t) \bar{\eta}_{\mathfrak{a}\mathfrak{c}}(m, t) \eta_{\mathfrak{b}\mathfrak{c}}(n, t)\,\mathrm{d}t \\ &+ \sum_{k=1}^{\infty} i^{2k} N_f(2k-1) \bar{\psi}_{\mathfrak{a}j2k}(m) \psi_{\mathfrak{b}j2k}(n).\end{aligned}$$

Remark 4.1. We remark that Theorem 4.1 coincides with [24, Theorem 2.3, p.64]. Since in the latter, the Neumann series part ([24, (2.2.6), p.51]) is replaced by [24, (2.2.9), p.51], it has a seemingly different outlook. On [24, p.92], it is claimed that the Neumann series expansion due to Titchmarsh and others ([30]) is dispensed with in his argument. One of our objectives is to show that the Sears-Titchmarsh inversion is imbedded in the process of replacing the sum involving W-function.

5. Treatment in GL_2

5.1. Unfolding method

Definition 5.1.

$$G = PGL_2(\mathbb{R}),$$

$$N = \left\{ \underline{n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \Big| n \in \mathbb{R} \right\}, \quad A = \left\{ \underline{c} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \Big| c \in \mathbb{R}^{\times} \right\}, \tag{5.1}$$

$$w = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is called the Weyl element.

$B = NA$ is the Borel subgroup of G. The Bruhat decomposition reads

$$G = B \cup NwB. \tag{5.2}$$

Although mainly we are concerned with subgroups of $SL_2(\mathbb{R})$, we state some results for a discrete subgroup Γ of G such that $\Gamma\backslash G$ is non-compact but has finite invariant volume. We may assume that

$$\Gamma \cap B = \Gamma \cap N = \Gamma_\infty,$$

where Γ_∞ is given by (2.5). Let ψ be a non-trivial character of $\mathbb{Z}\backslash\mathbb{R}$. Since

$$\Gamma_\infty \simeq \mathbb{Z} \subset \mathbb{R} \simeq N, \quad \Gamma_\infty\backslash N \simeq \mathbb{Z}\backslash\mathbb{R} \tag{5.3}$$

we may regard ψ as a character on $\Gamma_\infty\backslash N$.

Let $S(N\backslash G;\psi)$ denote the set of smooth functions f on G such that

$$f(\underline{n}g) = \psi(\underline{n})f(g), \quad \underline{n} \in N, g \in G, \tag{5.4}$$

where ψ is a character on $N\backslash G$. For $f \in S(N\backslash G;\psi)$ we define the **Poincaré series**

$$P_f(g) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} f(\gamma g) \tag{5.5}$$

the series being absolutely and uniformly convergent e.g. if f is a Schwarz function as defined in the following definition.

Definition 5.2. We introduce the norm on $N\backslash G$

$$\|g\|_N = \inf_{\underline{n} \in N} \|\underline{n}g\|.$$

A function $f \in S(N\backslash G;\psi)$ is said to be rapidly decreasing mod N if for any $M > 0$, there exists a $c_M > 0$ such that

$$|f(g)| < c_M \|g\|_N^{-M}.$$

A function $f \in S(N\backslash G;\psi)$ is called a Schwarz function mod N if all of its right translations are rapidly decreasing mod N. The space of all Schwarz function mod N is denoted $\mathcal{S}(N\backslash G;\psi)$.

Proposition 5.1. *If f is a Schwarz function mod N, then P_f is absolutely and uniformly convergent on compact subsets and $P_f \in S(N\backslash G;\psi)$.*

Definition 5.3. For $\psi \in S(\Gamma\backslash G)$ smooth, we define the **ψ-Whittaker function** (on G)

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{\Gamma_\infty\backslash N} \varphi(\underline{n}g)\psi^{-1}(\underline{n})\,\mathrm{d}\underline{n} \tag{5.6}$$

In view of (5.3), this corresponds to taking the Fourier coefficients.

Lemma 5.1. *Suppose* $\Gamma_2 \subset \Gamma_1 \subset \Gamma_0$ *and let*

$$\Gamma_2\backslash\Gamma_1 = \{\Gamma_2 s_\mu | \mu \in \mathrm{M}\} = \{s_\mu \Gamma_2\}$$

and

$$\Gamma_1\backslash\Gamma_0 = \{\Gamma_1 z_\nu | \nu \in \mathrm{N}\} = \{z_\nu \Gamma_1\}$$

be a complete set of representatives of equivalence classes. Then

$$\Gamma_0 = \cup_{\nu\in\mathrm{N}}\Gamma_1 z_\nu = \cup_\nu \cup_\mu \Gamma_2 s_\mu z_\nu = \cup_\lambda \Gamma_2 z_\lambda$$

i.e.

$$\Gamma_2\backslash\Gamma_1 \cup \Gamma_1\backslash\Gamma_0 = \Gamma_2\backslash\Gamma_0.$$

We apply this to the case $\Gamma_0 = \mathcal{H}$ and $\Gamma_2 = \Gamma_\infty$.

Proposition 5.2. *For* $f \in \mathcal{S}(N\backslash G;\psi)$, $\varphi \in S(\Gamma\backslash G)$, *we have*

$$(\varphi, P_f) = \int_{N\backslash G} \bar{f}(g) W_{\varphi,\psi}(g)\,\mathrm{d}g. \tag{5.7}$$

Proof. Substituting the definition (5.4), we obtain

$$\begin{aligned}(\varphi, P_f) &= \int_{\Gamma\backslash G} \varphi(g)\overline{P_f(g)}\,\mathrm{d}g = \int_{\Gamma\backslash G}\varphi(g)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{f(\gamma g)}\,\mathrm{d}g\\ &= \int_{\Gamma_\infty\backslash G}\varphi(g)\overline{f(g)}\,\mathrm{d}g\\ &= \int_{N\backslash G}\int_{\Gamma_\infty\backslash N}\varphi(\underline{n}g)\overline{f(\underline{n}g)}\,\mathrm{d}g\mathrm{d}\underline{n}\end{aligned}$$

by Lemma 5.1. Applying (5.4), we may rewrite the double integral into a repeated integral

$$\int_{N\backslash G}\bar{f}(g)\,\mathrm{d}g\int_{\Gamma_\infty\backslash N}\bar{\psi}(\underline{n})\varphi(\underline{n}g)\,\mathrm{d}\underline{n}$$

whose inner integral is $W_{\varphi,\psi}(g)$ in (5.6), whence the result follows. □

5.2. Double coset decomposition

In the light of (5.3), the inner integral in (5.7) corresponds to the 0th Fourier coefficient.

We decompose Γ according to the Bruhat decomposition (5.2) of G. Let

$$\Gamma_c = Nw\underline{c}N,$$

where $\underline{c}$ is from (5.1) and let

$$\Omega(\Gamma) = \{c \in \mathbb{R}^\times | \Gamma_c \neq 0\}.$$

For each $\gamma \in \Gamma_c$, we have the decomposition

$$\gamma = n_1(\gamma)w\underline{c}n_2(\gamma)$$

with $n_j(\gamma) \in N$. Since

$$\Gamma = \Gamma_\infty \bigcup_{c\in\Omega(\Gamma)} \Gamma_c,$$

we have the double coset decomposition

$$\Gamma_\infty\backslash\Gamma = \{1\} \bigcup_{c\in\Omega(\Gamma)} \Gamma_\infty\backslash\Gamma_c.$$

Definition 5.4. For $c \in \Omega(\Gamma)$ and non-trivial additive characters ψ_1, ψ_2 on $\Gamma_\infty\backslash N$, let

$$S_\Gamma(c, \psi_1, \psi_2) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty} \psi_1(n_1(\gamma))\psi_2(n_2(\gamma)). \tag{5.8}$$

called the **Kloostermann sum**. In the above setting we may define an intertwining operator $K(c;\psi_1,\psi_2) : S(N\backslash G;\psi_1) \to S(N\backslash G;\psi_2)$ given by

$$K(c;\psi_1,\psi_2)(f)(g) = \int_N f(w\underline{cng})\psi_2^{-1}(\underline{n})\,\mathrm{d}\underline{n}.$$

6. Maass forms and Poincaré series

6.1. Double coset decomposition

Lemma 6.1. *Let $\mathfrak{a}, \mathfrak{b}$ be cusps of Γ.*

Then we have a disjoint union

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a}\mathfrak{b}}\Omega_\infty \bigcup_{c>0} \bigcup_{d(\mathrm{mod} c)} \Gamma_\infty\gamma_{d/c}\Gamma_\infty,$$

where c, d run over integers such that

$$\gamma_{d/c} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}. \tag{6.1}$$

If $g : PSL(2,\mathbb{R}) \to \mathbb{C}$ and

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} g(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}})$$

is absolutely convergent, then

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} g(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}) = \delta_{\mathfrak{a}\mathfrak{b}}g(1) + \sum_{c>0}\sum_{d(\mathrm{mod} c)}\sum_{n\in\mathbb{Z}} g(\gamma_{d/c}T^n), \tag{6.2}$$

where c, d are subject to condition (6.1). Or if $f(z)$ is of sufficiently rapid decay on $\mathcal{H}$, then

$$\sum_{\gamma\in\Gamma} f(\gamma z)$$

is automorphic and

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} f(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z) = \sum_{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} f(\gamma z)$$

$$= \delta_{\mathfrak{a}\mathfrak{b}} f(z) + \sum_{c>0} \sum_{d (\mathrm{mod} c)} \sum_{n \in \mathbb{Z}} f(\gamma_{d/c}(z+n))$$

under the same condition on c, d.

Theorem 6.1. *For the general Poincaré series* (4.6)*, we have the Fourier expansion*

$$E_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z|\psi) = \delta_{\mathfrak{a}\mathfrak{b}} e^{2\pi i m z} \psi(y) + \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \sum_{c=1}^{\infty} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) I(m,n,c,y),$$

where in analogy to (3.9)

$$\begin{aligned} I(m,n,c,y) &= \int_{-\infty}^{\infty} \psi\left(\mathrm{Im}\, \frac{c^{-2}}{u+iy}\right) e^{2\pi i \frac{-mc^{-2}}{u+iy} - 2\pi i n u} \,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} \psi\left(\frac{-yc^{-2}}{u^2+y^2}\right) e^{2\pi i \frac{-mc^{-2}}{u+iy} - 2\pi i n u} \,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} \psi\left(\frac{-c^{-2}}{y(1+\xi^2)}\right) e^{2\pi i \frac{-mc^{-2}}{y(\xi+i)} - 2\pi i n y \xi} y \,\mathrm{d}\xi. \end{aligned}$$

We concentrate on the case

$$P_{\mathfrak{a}m}(z,s) = E_{\mathfrak{a}m}(z|y^s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \left(\mathrm{Im}\, \sigma_{\mathfrak{a}}^{-1} \gamma z\right)^s e^{2\pi i m (\sigma_{\mathfrak{a}}^{-1} \gamma z)}, \tag{6.3}$$

for which we have

Corollary 6.1. *We have the Fourier expansion*

$$P_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a}\mathfrak{b}} y^s e^{2\pi i m z} + \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \sum_{c=1}^{\infty} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) I(m,n,c,y), \tag{6.4}$$

where in analogy to (3.9)

$$I(m,n,c,y) = \int_{-\infty}^{\infty} (-1)^s c^{-2s} (1+\xi^2)^{-s} e^{2\pi i \frac{-mc^{-2}}{y(\xi+i)} - 2\pi i n y \xi} y^{-s+1} \mathrm{d}\xi. \tag{6.5}$$

6.2. Unfolding method

By a similar reasoning to the proof of Proposition 5.2, we have

Theorem 6.2. *Let f be automorphic, Lebesgue measurable over $\Gamma \backslash \mathcal{H}$ and ψ is a measurable function on the positive reals. Then*

$$\langle f, E_{\mathfrak{a}n}(\cdot|\psi) \rangle = \int_0^{\infty} \int_0^1 f(\sigma_{\mathfrak{a}} w) \bar{\psi}(\mathrm{Im}\, w) e^{-2\pi i \bar{w} n} \,\mathrm{d}\mu(w) \tag{6.6}$$

which often reduces to

$$\langle f, E_{\mathfrak{a}n}(\cdot|\psi) \rangle = \int_0^{\infty} f_{\mathfrak{a}}(y) \bar{\psi}(y) e^{-2\pi i n \bar{z}} \frac{\mathrm{d}y}{y^2}, \tag{6.7}$$

where $f_{\mathfrak{a}}(y) = f_{\mathfrak{a}0}(y)$ is the 0th coefficient (4.3).

Proof.

$$\langle f, E_{\mathfrak{a}n}(\cdot|\psi)\rangle = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\Gamma\backslash\mathcal{H}} f(z)\bar{\psi}(\operatorname{Im}\sigma_{\mathfrak{a}}^{-1}z)e^{-2\pi in\overline{\operatorname{Im}\sigma_{\mathfrak{a}}^{-1}z}}\,\mathrm{d}\mu(z).$$

Make the change of variable $w = \sigma_{\mathfrak{a}}^{-1}\gamma z$ and use the invariance of μ to deduce that

$$\begin{aligned}\langle f, E_{\mathfrak{a}n}(\cdot|\psi)\rangle &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma(\Gamma\backslash\mathcal{H})} f(\gamma^{-1}\sigma_{\mathfrak{a}}w)\bar{\psi}(\operatorname{Im}w)e^{-2\pi in\bar{w}}\,\mathrm{d}\mu(w)\\ &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma(\Gamma\backslash\mathcal{H})} f(\sigma_{\mathfrak{a}}w)\bar{\psi}(\operatorname{Im}w)e^{-2\pi in\bar{w}}\,\mathrm{d}\mu(w).\end{aligned}$$

by automorphy of f. Now as γ runs over $\Gamma_{\mathfrak{a}}\backslash\Gamma$, $\sigma_{\mathfrak{a}}^{-1}\gamma(\Gamma\backslash\mathcal{H})$ covers the strip $\Gamma_\infty\backslash\mathcal{H}$, whence

$$\langle f, E_{\mathfrak{a}n}(\cdot|\psi)\rangle = \int_0^\infty\int_0^1 f(\sigma_{\mathfrak{a}}w)\bar{\psi}(\operatorname{Im}w)e^{-2\pi i\bar{w}n}\,\mathrm{d}\mu(w),$$

whence (6.6) follows. □

Specifying $E_{\mathfrak{a}n}(\cdot|\psi)$ to $P_{\mathfrak{b}n}(\cdot,\overline{s_2})$ in (6.3), (6.6) reads

$$\langle f, P_{\mathfrak{b}n}(\cdot,\overline{s_2})\rangle = \int_0^\infty\int_0^1 f(\sigma_{\mathfrak{b}}w)y^{s_2}e^{-2\pi in\bar{z}}\,\frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{6.8}$$

Below we apply (6.8) with $f(\sigma_{\mathfrak{b}}z) = P_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}z, s)$ in (6.4):

$$P_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{a}\mathfrak{b}}y^s e^{2\pi imz} + \sum_{l=-\infty}^{\infty} e^{2\pi ilx}\sum_{c=1}^{\infty} S_{\mathfrak{a}\mathfrak{b}}(m,l;c)I(m,l,c,y), \tag{6.9}$$

where

$$I(m,l,c,y) = \int_{-\infty}^{\infty}(-1)^s c^{-2s}(1+\xi^2)^{-s}e^{2\pi i\frac{-mc^{-2}}{y(\xi+i)}-2\pi ily\xi}\,y^{-s+1}\mathrm{d}\xi.$$

Substituting the equalities in Corollary 6.1, we note that the first term in (6.9) contributes

$$\delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}\int_0^\infty e^{-2\pi(m+n)y}y^{s_1+s_2-1}\,\frac{\mathrm{d}y}{y}$$

which gives the first term in (6.10) below. The second term in (6.9) contributes the second term in (6.10). Thus we have the following

Lemma 6.2. *For cusps* $\mathfrak{a},\mathfrak{b}$ *we have*

$$\begin{aligned}\langle P_{\mathfrak{a}m}(\cdot,s_1), P_{\mathfrak{b}n}(\cdot,\overline{s_2})\rangle &= \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}\Gamma(s_1+s_2-1)(4\pi m)^{1-s_1-s_2}\\ &\quad+\int_0^\infty y^{s_2-s_1-1}\sum_{c=1}^{\infty}c^{-2s_1}S_{\mathfrak{a}\mathfrak{b}}(m,n;c)J(m,n,c,y)\,\mathrm{d}y,\end{aligned} \tag{6.10}$$

where

$$J(m,n,c,y) = \int_{-\infty}^{\infty}(1+\xi^2)^{-s_1}e^{2\pi\frac{-mc^{-2}}{y(1-i\xi)}-2\pi iny\xi}\,\mathrm{d}\xi. \tag{6.11}$$

7. Proof of Kuznetsov sum formula

7.1. Neumann series part

From Lemma we may follow the argument of Motohashi [24, pp.45-47] to prove

Theorem 7.1. *For $c>0$, $\operatorname{Re} s_2 + c > \operatorname{Re} s_1 > c + \frac{1}{4}$ we have*

$$\begin{aligned}&\langle P_{\mathfrak{a}m}(\cdot, s_1), P_{\mathfrak{b}n}(\cdot, \overline{s_2})\rangle = \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}\Gamma(s_1+s_2-1)(4\pi m)^{1-s_1-s_2} \\ &+ 2^{2(1-s_2)}\pi^{s_1-s_2+1}n^{s_1-s_2}\Gamma(s_1+s_2-1) \\ &\times \sum_{c=1}^{\infty} c^{-2s_1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) G_{1,3}^{2,0}\left(\frac{4\pi^2 mn}{c^2} \middle| \begin{matrix} s_1 \\ 0, s_2-s_1, 1-s_1 \end{matrix} \right)\end{aligned} \tag{7.1}$$

where

$$G_{1,3}^{2,0}\left(z \middle| \begin{matrix} s_1 \\ 0, s_2-s_1, s_3 \end{matrix} \right) = \frac{1}{2\pi i}\int_c \frac{\Gamma(w)\Gamma(w+s_2-s_1)}{\Gamma(w+s_2)\Gamma(1-s_3-w)} z^{-s}\,\mathrm{d}w. \tag{7.2}$$

Lemma 7.1.

$$\begin{aligned}&\frac{\Gamma(s)}{\Gamma(1-s)} G_{1,3}^{2,0}\left(z \middle| \begin{matrix} 1 \\ 0, s-1, 0 \end{matrix} \right) \\ &= z^{s-1} - \frac{1}{\sqrt{z}} \sum_{k=1}^{\infty} (2k-1)\frac{\Gamma(k-1+s)}{\Gamma(k+1-s)} J_{2k-1}\left(2\sqrt{z}\right)\end{aligned} \tag{7.3}$$

(*Motohashi* [24, (2.4.3), p.63]) *and*

$$\frac{\Gamma(s)}{\Gamma(1-s)} G_{1,3}^{2,0}\left(z \middle| \begin{matrix} 0 \\ 0, 0, -1 \end{matrix} \right) = \frac{1}{\sqrt{z}} J_1\left(2\sqrt{z}\right).$$

$$G_{1,2}^{2,0}\left(z \middle| \begin{matrix} 0 \\ b, c \end{matrix} \right) = z^{\frac{1}{2}(b+c-1)} e^{-\frac{1}{2}z} W_{k,m}(z),$$

where $k = \frac{1}{2}(1+b+c), m = \frac{1}{2}(b-c)$.

$$W_{0,\mu}(z) = \frac{1}{\sqrt{\pi}}\sqrt{z}K_\mu\left(\frac{1}{2}z\right)$$

which is [9, p.265]*. Hence*

$$G_{1,2}^{2,0}\left(z\,\middle|\,\begin{matrix}0\\ \eta-\frac{1}{2},-\eta-\frac{1}{2}\end{matrix}\right)=\frac{1}{\sqrt{\pi}}z^{-\frac{1}{2}}e^{-\frac{1}{2}z}K_\eta\left(\frac{1}{2}z\right). \tag{7.4}$$

Another formula used is

$$G_{0,2}^{2,0}\left(z\,\middle|\,\begin{matrix}-\\ a,b\end{matrix}\right)=\frac{1}{2\pi i}\int_{(c)}\Gamma(a+s)\Gamma(b+s)z^{-s}\,\mathrm{d}s=2\,z^{\frac{1}{2}(a+b)}K_{a-b}\big(2\sqrt{z}\,\big). \tag{7.5}$$

This is applied in the special case

$$G_{0,2}^{2,0}\left(z\,\middle|\,\begin{matrix}-\\ -\frac{1}{2}+ir,-\frac{1}{2}-ir\end{matrix}\right)=2\,z^{-\frac{1}{2}}K_{2ir}\big(2\sqrt{z}\,\big).$$

For the proof of Theorem 4.3 we need two more well-known formulas

$$K_\nu(z)=\frac{\pi}{2\sin(\pi\nu)}\left(I_{-\nu}(z)-I_\nu(z)\right)$$

and

$$I_\nu(z)=e^{-\frac{\pi i\nu}{2}}J_\nu(iz),\quad -\frac{\pi}{2}<\arg z<\frac{\pi}{2} \tag{7.6}$$

We shall prove Theorem 4.3 by modifying Motohashi's argument [24, pp.44-67] and under the assumption that f is a Mellin transform of f^*:

$$f^*(s)=\int_0^\infty f(x)\Big(\frac{x}{2}\Big)^{-2s}\mathrm{d}x$$

or

$$f(x)=\frac{1}{2\pi i}\int_{(\alpha)}f^*(x)\Big(\frac{x}{2}\Big)^{2s-1}\mathrm{d}s. \tag{7.7}$$

We use the special case of Theorem 7.1 with $s_1=1, s_2=s$ in conjunction with (7.3) in the following winding manner. We apply the Mellin inversion formula (7.7) to

$$\begin{aligned}&\langle P_{\mathfrak{a}m}(\cdot,1),P_{\mathfrak{b}n}(\cdot,\overline{s})\rangle=\delta_{\mathfrak{ab}}\delta_{mn}\Gamma(s)(4\pi m)^{-s}\\ &+\pi(4\pi n)^{1-s}\sum_{c=1}^\infty c^{-2}S_{\mathfrak{ab}}(m,n;c)\Gamma(s)G_{1,3}^{2,0}\left(\frac{4\pi^2mn}{c^2}\,\middle|\,\begin{matrix}1\\ 0,s-1,0\end{matrix}\right).\end{aligned} \tag{7.8}$$

Multiplying (7.3) with $z = \frac{4\pi^2 mn}{c^2}$ by $c^{-2}S_{\mathfrak{a}\mathfrak{b}}(m,n;c)\frac{1}{\Gamma(1-s)}$ and summing over $c = 1, 2, \cdots$, we obtain

$$\sum_{c=1}^{\infty} c^{-2} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) G^{2,0}_{1,3}\left(\frac{4\pi^2 mn}{c^2} \middle| \begin{matrix} 1 \\ 0, s-1, 0 \end{matrix} \right) \frac{\Gamma(s)}{\Gamma(1-s)} \tag{7.9}$$
$$= (2\pi\sqrt{mn})^{2s-2} \sum_{c=1}^{\infty} c^{-2s} S_{\mathfrak{a}\mathfrak{b}}(m,n;c)$$
$$- \frac{1}{2\pi\sqrt{mn}} \sum_{c=1}^{\infty} c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \sum_{k=1}^{\infty} (2k-1) \frac{\Gamma(k-1+s)}{\Gamma(k+1-s)} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Then we substitute (7.8) to replace the right-hand side of (7.9) by

$$\frac{1}{\pi}(4\pi n)^{s-1} \frac{1}{\Gamma(1-s)} \langle P_{\mathfrak{a}m}(\cdot, 1), P_{\mathfrak{b}n}(\cdot, \overline{s}) \rangle - \delta_{\mathfrak{a}\mathfrak{b}} \delta_{mn} \frac{1}{4\pi^2 n} \left(\frac{n}{m}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)}$$

to deduce that

$$2\sqrt{mn}(4\pi n)^{s-1} \frac{1}{\Gamma(1-s)} \langle P_{\mathfrak{a}m}(\cdot, 1), P_{\mathfrak{b}n}(\cdot, \overline{s}) \rangle - \delta_{\mathfrak{a}\mathfrak{b}} \delta_{mn} \frac{1}{2\pi} \left(\frac{n}{m}\right)^{s-1/2} \frac{\Gamma(s)}{\Gamma(1-s)}$$
$$= (2\pi\sqrt{mn})^{2s-1} \sum_{c=1}^{\infty} c^{-2s} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \tag{7.10}$$
$$- \sum_{c=1}^{\infty} c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \sum_{k=1}^{\infty} (2k-1) \frac{\Gamma(k-1+s)}{\Gamma(k+1-s)} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

7.2. Spectral side

Then we replace the inner product part by the spectral expression in Lemma 7.2. To state it we prepare standard notation.

Let $\mathfrak{a}$ be a cusp of Γ. Let $\{u_j(z) : j \geq 0\}$ be a complete orthonormal system of Maass forms and let $\{E_{\mathfrak{c}}(z,s) : s = \frac{1}{2} + it, t \in \mathbb{R}\}$ be the eigen packet of Eisenstein series in $\mathcal{L}(\Gamma\backslash\mathcal{H})$ ([12, p.117]) and

$$u_j(\sigma_{\mathfrak{a}} z) = \rho_{\mathfrak{a}j}(0) y^{1-s_j} + \sum_{n \neq 0} \rho_{\mathfrak{a}j}(n) W_{0,s_j}(nz) \tag{7.11}$$
$$= \rho_{\mathfrak{a}j}(0) y^{1-s_j} + 2\sqrt{y} \sum_{n \neq 0} \rho_{\mathfrak{a}j}(n) K_{s_j-1/2}(2\pi n y) e(nx),$$

$$E_{\mathfrak{c}}(\sigma_{\mathfrak{a}} z, s) = \delta_{\mathfrak{a}\mathfrak{c}} y^s + \varphi_{\mathfrak{a}\mathfrak{c}}(s) y^{1-s} + \sum_{n \neq 0} \varphi_{\mathfrak{a}\mathfrak{c}}(n,s) W_s(nz) \tag{7.12}$$
$$= \delta_{\mathfrak{a}\mathfrak{c}} y^s + \varphi_{\mathfrak{a}\mathfrak{c}}(s) y^{1-s} + 2\sqrt{y} \sum_{n \neq 0} \varphi_{\mathfrak{a}\mathfrak{c}}(n,s) K_{s-1/2}(2\pi |n| y) e(nx)$$

are their Fourier expansions with Fourier coefficients $\rho_{\mathfrak{a}j}(n)$ and $\varphi_{\mathfrak{ac}}(n,s)$, where

$$\varphi_{\mathfrak{ac}}(n,s)=\pi^s|n|^{s-1}\frac{1}{\Gamma(s)}\sum_{c\in C_{\mathfrak{ac}}}c^{-2s}S_{\mathfrak{ac}}(0,n;c).$$

Hence, in particular,

$$\varphi_{\infty\infty}(n,s)=\frac{\pi^s}{\Gamma(s)\zeta(2s)}|n|^{-1/2}\sum_{ab=|n|}\left(\frac{a}{b}\right)^{s-1/2}=\frac{\pi^s}{\Gamma(s)\zeta(2s)}|n|^{s-1}\sigma_{1-2s}(|n|)$$

where σ indicates the sum-of-divisors function, [12, (3.25), p.67].

On [12, p.118], the normalization is introduced, which we will use in this paper:

$$\nu_{\mathfrak{a}j}(n)=\left(\frac{4\pi|n|}{\cosh\pi t_j}\right)^{1/2}\rho_{\mathfrak{a}j}(n),\quad \eta_{\mathfrak{ac}}(n,t)=\left(\frac{4\pi|n|}{\cosh\pi t}\right)^{1/2}\varphi_{\mathfrak{ac}}(n,1/2+it)\tag{7.13}$$

for $n\neq 0$ and eigenvalues $s_j=it_j$.

Lemma 7.2. (Parseval formula), [24, Lemma 2.2] *Let* $\{u_j\}$ *be as in* (7.11)*. Then for* $\operatorname{Re}s_j>\frac{3}{4}$, $j=1,2$ *we have*

$$\begin{aligned}\langle P_{\mathfrak{a}m}(\cdot,s_1),P_{\mathfrak{b}n}(\cdot,\overline{s_2})\rangle&=\frac{\pi}{\Gamma(s_1)\Gamma(s_2)}(4\pi\sqrt{mn})^{1-s_1-s_2}\left(\frac{n}{m}\right)^{\frac12(s_1-s_2)}\\&\times\Bigg[\sum_{j=1}^{\infty}\overline{\rho_{\mathfrak{a}j}(m)}\rho_{\mathfrak{a}j}(n)\Theta(s_1,s_2;s_j)\\&\quad+\frac{1}{4\pi}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}\bar\varphi_{\mathfrak{ac}}\left(m,\frac12+it\right)\varphi_{\mathfrak{bc}}\left(n,\frac12+it\right)\Theta(s_1,s_2;t)\mathrm{d}t\Bigg]\end{aligned}\tag{7.14}$$

where

$$\begin{aligned}&\Theta(s_1,s_2;r)\\&=\Gamma\left(s_1-\frac12+ir\right)\Gamma\left(s_1-\frac12-ir\right)\Gamma\left(s_2-\frac12+ir\right)\Gamma\left(s_2-\frac12-ir\right).\end{aligned}\tag{7.15}$$

In particular

$$\begin{aligned}\langle P_{\mathfrak{a}m}(\cdot,1),P_{\mathfrak{a}n}(\cdot,\overline{s})\rangle&=\frac{\pi}{\Gamma(s)}(4\pi\sqrt{mn})^{-1-s}\left(\frac{n}{m}\right)^{\frac12(1-s)}\\&\times\Bigg[\sum_{j=1}^{\infty}\overline{\nu_{\mathfrak{a}j}(m)}\nu_{\mathfrak{a}j}(n)\Gamma\left(s-\frac12+is_j\right)\Gamma\left(s-\frac12-is_j\right)\\&\quad+\frac{1}{4\pi}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}\bar\eta_{\mathfrak{ac}}(m,r)\eta_{\mathfrak{bc}}(n,r)\Gamma\left(s-\frac12+it\right)\Gamma\left(s-\frac12-it\right)\mathrm{d}r\Bigg]\end{aligned}\tag{7.16}$$

Proof. The Parseval formula reads

$$\begin{aligned}\langle P_{\mathfrak{a}m}(\cdot,s_1),P_{\mathfrak{b}n}(\cdot,\overline{s_2})\rangle&=\sum_{j=0}^{\infty}\langle P_{\mathfrak{a}m}(\cdot,s_1),u_j\rangle\overline{\langle P_{\mathfrak{b}n}(\cdot,\overline{s_2}),u_j\rangle}\\&+\frac{1}{4\pi}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}\left\langle P_{\mathfrak{a}m}(\cdot,s_1),E_{\mathfrak{c}}\left(\cdot,\frac12+it\right)\right\rangle\overline{\left\langle P_{\mathfrak{b}n}(\cdot,\overline{s_2}),E_{\mathfrak{c}}\left(\cdot,\frac12+it\right)\right\rangle}\mathrm{d}t.\end{aligned}$$

Specifying $E_{\mathfrak{a}n}(\cdot|\psi)$ to $P_{\mathfrak{a}m}(\cdot,s)$ in (6.3), (6.7) reads

$$\langle P_{\mathfrak{a}m}(\cdot,s),f\rangle = \int_0^\infty \overline{f_{\mathfrak{a}}(y)}y^{s-2}e^{-2\pi ny}\,\mathrm{d}y, \tag{7.17}$$

In the case u_j, we use (7.11) to compute the integral $\overline{u_{\mathfrak{a}i}(y)}$ to be δ_{mn}, so that (7.17) amounts to

$$\begin{aligned}\langle P_{\mathfrak{a}m}(\cdot,s),u_j\rangle &= \overline{\rho_{\mathfrak{a}j}(m)}\int_0^\infty y^{s-\frac{3}{2}}e^{-2\pi my}K_{s_j-1/2}(2\pi my)\,\mathrm{d}y\\ &= (2\pi m)^{\frac{1}{2}-s}\overline{\rho_{\mathfrak{a}j}(m)}\int_0^\infty y^{s-\frac{3}{2}}e^{-y}K_{it_j}(y)\,\mathrm{d}y\\ &= \sqrt{\pi}(4\pi m)^{\frac{1}{2}-s}\overline{\rho_{\mathfrak{a}j}(m)}\frac{\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma(s)}\end{aligned}$$

by the Mellin transform of (7.4):

$$\int_0^\infty y^{s-\frac{3}{2}}e^{-y}K_\eta(y)\,\mathrm{d}y = \sqrt{2\pi}2^{-s}\frac{\Gamma\left(s+\eta-\frac{1}{2}\right)\Gamma\left(s-\eta-\frac{1}{2}\right)}{\Gamma(s)}.$$

$$\left\langle P_{\mathfrak{a}m}(\cdot,s),E_{\mathfrak{c}}\left(\cdot,\frac{1}{2}+it\right)\right\rangle = \int_0^\infty \overline{E_{\mathfrak{c}}(y)}y^{s-2}e^{-2\pi ny}\,\mathrm{d}y \tag{7.18}$$

In this case we use (7.12) and the integral $\overline{E_{\mathfrak{c}i}(y)}$ to be δ_{mn}, so that (7.18) amounts to

$$\begin{aligned}&\left\langle P_{\mathfrak{a}m}(\cdot,s),E_{\mathfrak{c}}\left(\cdot,\frac{1}{2}+it\right)\right\rangle\\ &= \varphi_{\mathfrak{a}\mathfrak{c}}\left(m,\frac{1}{2}+it\right)\int_0^\infty y^{s-\frac{3}{2}}e^{-2\pi my}K_{-it}(2\pi my)\,\mathrm{d}y\\ &= \sqrt{\pi}(4\pi m)^{\frac{1}{2}-s}\varphi_{\mathfrak{a}\mathfrak{c}}\left(m,\frac{1}{2}+it\right)\frac{\Gamma\left(s-\frac{1}{2}+it\right)\Gamma\left(s-\frac{1}{2}-it\right)}{\Gamma(s)}.\end{aligned}$$

(7.16) follows in view of

$$\Theta(1,s;r) = \frac{\pi\Gamma\left(s-\frac{1}{2}+ir\right)\Gamma\left(s-\frac{1}{2}-ir\right)}{\cosh\pi r}.$$

□

7.3. Mellin invesion

$$\begin{aligned}&(4\pi)^{-2}(mn)^{-1}\frac{\sin\pi s}{\pi}\left[\sum_{j=1}^\infty\overline{\nu_{\mathfrak{a}j}(m)}\nu_{\mathfrak{a}j}(n)\Gamma\left(s-\frac{1}{2}+is_j\right)\Gamma\left(s-\frac{1}{2}-is_j\right)\right.\\ &\left.+\frac{1}{4\pi}\sum_{\mathfrak{c}}\int_{-\infty}^\infty\bar{\eta}_{\mathfrak{a}\mathfrak{c}}(m,r)\eta_{\mathfrak{b}\mathfrak{c}}(n,r)\Gamma\left(s-\frac{1}{2}+it\right)\Gamma\left(s-\frac{1}{2}-it\right)\mathrm{d}r\right]\\ &-\delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}\frac{1}{4\pi^2 n}\left(\frac{n}{m}\right)^s\frac{\Gamma(s)}{\Gamma(1-s)}\end{aligned}$$

$$= (2\pi\sqrt{mn})^{2s-2} \sum_{c=1}^{\infty} c^{-2s} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \tag{7.19}$$
$$- \frac{1}{2\pi\sqrt{mn}} \sum_{c=1}^{\infty} c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \sum_{k=1}^{\infty} (2k-1) \frac{\Gamma(k-1+s)}{\Gamma(k+1-s)} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

To prove Theorem 4.3, we apply (7.7), i.e. multiplying (7.19) by $f^*(x)\left(\frac{x}{2}\right)^{2s-1}$ and integrate along $\sigma = \alpha$.

Thereby we interchange the order of integration and appeal to the known formulas for G-functions (7.5), (7.13), and (7.6) successively for the integrals of the form

$$\int_{(\alpha)} \Gamma\left(s - \frac{1}{2} + ir\right) \Gamma\left(s - \frac{1}{2} - ir\right) z^{-s}\, \mathrm{d}r$$

and we appeal to (7.5) for integrals of the form

$$\int_{(\alpha)} \frac{\Gamma(k-1+s)}{\Gamma(k+1-s)} z^{-s}\, \mathrm{d}r.$$

This winding manner occurs because we avoid the use of Sears-Titchmarsh inversion and indeed, the f^{∞} part in Theorem 4.2 does not appear explicitly and embedded in the replacement process.

tool	Double coset, Poisson	Inn. prod. unfolding	appl.
holom.	Fourier exp.	Petersson	$\circ$
non-hol.	Fourier exp.	Kuznetsov	$\circ$

Table 3. Relation between holomorphic and non-holomorphic trace formula

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