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ON THE SOLVABILITY OF TIME CONFORMABLE FRACTIONAL EQUATION SET ON SINGULAR DOMAIN OF $\mathbb{R}^{N+1}$

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Abstract In this paper, we investigate the solvability of time conformable fractional equation set in a singular cylindrical domain in $\mathbb{R}^{N+1}$. Some regularity results are obtained for the classical solutions by using the Dunford operational calculus

Keywords Hölder spaces, Abstract differential equations, Strict solution, Conformable fractional derivative.

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1. Introduction and motivation

First, let Π be the cylindrical defined by

$$\Pi = [0,1] \times \Omega(t),$$

where $\Omega(t)$ is the singular domain given by

$$\Omega(t) = \left\{ (x_1, x_2, .., x_N) \in \mathbb{R}^N : \sqrt{x_1^2 + ... + x_N^2} \leq \varphi(t) \right\}.$$

Here φ is the functions of parametrization satisfying

$$\varphi(0) = 0,$$

and

$$\varphi(t) > 0, \ t \in]0,1].$$

In the cylindrical domain Π, we consider the following linear conformable fractional differential equation

$$\mathcal{D}_t^{\alpha} u(t,x) + \sum_{j=1}^{N} D_{x_i}^{2m} u(t,x) = h(t,x), \ \alpha \in (0,1], \ m \in \mathbb{N}^*, \tag{1.1}$$

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where $x = (x_1, x_2, .., x_N)$ denotes a generic point of $\mathbb{R}^N$ and $\mathcal{D}_t^\alpha$ is the standard conformable time fractional derivative of order α in the sense stated in [1] and [10]. Recall here that for a given function f the conformable time fractional derivative of order $\alpha$is defined by

$$\mathcal{D}_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon},$$

For more information, we refer the reader to [3], [11], [12]. Furthermore, the relationship with the classic derivative is given by the following useful relation

$$\mathcal{D}_t^\alpha f = t^{1-\alpha} D_t f. \tag{1.2}$$

The conformable fractional derivatives can be viewed in some sense as an extension of the classical one and it is important to note here that this type of derivatives satisfies all concepts of ordinary calculus such as: product, quotient and chain rules, Rolle theorem and mean value theorem. Furthermore, the fractional conformable derivatives are used to better understand some physical and engineering systems. To be more precise, using the conformable derivatives calculus, an alternative representation of the diffusion equation is given to improve the modeling of anomalous diffusion and also to develop the Swartzendruber model for description of non-Darcian flow in porous media [13]. Before finishing, we just note that this work is motivated by the fact that there is a few results devoted to the study of conformable fractional boundary problems set on non smooth domains. The method of investigation, is inspired from [4], [5], [9] in where some concrete evolution problems were treated using an abstract point of view. This abstract approach is based essentially on the use of the well known Dunford's operational calculus combined with the techniques used in [2], [6], [7] and [8]. In this work, the right hand side f of (1.1) is a taken in

$$C^\theta([0,1]; C(\Omega)), 0 < \theta < 1,$$

denoting the space of the bounded and θ-Hölder continuous functions endowed with the norm

$$\|u\|_{C^\theta([0,1];C(\Omega))} = \max_{(t,x)\in\overline{\Pi}} |f(x)| + \sup_{(t',x')\neq(t,x)} \frac{\|f(x') - f(x)\|_{C(\Omega)}}{\|t' - t, x' - x\|^\theta}.$$

We suppose also that (1.1) is associated with the following initial and boundary conditions

$$u|_{\{0\}\times\Omega} = 0,\ u|_{\{1\}\times\Omega} = 0, \tag{1.3}$$

$$u|_{[0,1]\times\partial\Omega} = 0, \tag{1.4}$$

where $D(1, \varphi(1))$ is the disk of radius $\varphi(1)$ centered at $(1, 0)$. Our strategy is based on the approximation of the singular domain Π by a sequences of domains given by

$$\Pi_n = [t_n, 1] \times \Omega(t),$$

where $(t_n)_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers such

$$\lim_{n\to+\infty} t_n = 0.$$

Putting

$$u_n = u|_{\Pi_n}.$$

Consequently, the solution u of Problem (1.1) will be approached by the solutions u_n of

$$\mathcal{D}_t^{\alpha} u_n(t,x) - \sum_{j=1}^{N} \partial_{x_i}^{2m} u_n(t,x) = f(t,x), \ \ (t,x) \in \Pi_n, \tag{1.5}$$

subject to

$$u|_{\{t_n\}\times\Omega} = 0, \, u|_{\{1\}\times\Omega} = 0, \tag{1.6}$$

$$u|_{[t_n,1]\times\partial\Omega} = 0. \tag{1.7}$$

2. Results for approached problem (1.5)–(1.6)-(1.7)

We start with he abstract setting of the approached problem (1.5)-(1.7). For this reason, we consider the following change of variables

$$\begin{aligned} & T : \Pi_n \to Q_n, \\ & (t, x_1, x_2, x_3) \mapsto (t, \xi_1, .., \xi_N) = \left(t, \frac{x_1}{\varphi(t)}, ..., \frac{x_N}{\varphi(t)} \right), \end{aligned} \tag{2.1}$$

where

$$Q_n = [t_n, 1] \times D,$$

here

$$D := D(0,1) = \left\{ \xi = (\xi_1, .., \xi_N) \in \mathbb{R}^N : \sqrt{\xi_1^2 + ... + \xi_N^2} \le 1 \right\}. \tag{2.2}$$

Now, we introduce the following change of functions

$$\begin{cases} u_n(t, x_1, .., x_N) = v_n(t, \xi_1, .., \xi_N), \\ \text{and} \\ h(t, x_1, .., x_N) \ \ = g(t, \xi_1, .., \xi_N). \end{cases} \tag{2.3}$$

In the sequel, to make the notation less cluttered, we denote also by $\xi = (\xi_1, .., \xi_N)$ a generic point of $\mathbb{R}^N$. Keeping in mind properties (1.2), the new version of (1.5) is given by

$$t^{1-\alpha} D_t v_n(t,\xi) - \frac{1}{\varphi^{2m}(t)} \sum_{j=1}^{N} D_{\xi_i}^{2m} v_n(t,\xi) - \frac{\varphi'(t)}{\varphi(t)} \sum_{j=1}^{n} \xi_j D_{\xi_j} v_n(t,\xi) = f(t,\xi) \tag{2.4}$$

or

$$D_t v_n(t,\xi) - L\left(t, \xi, D_{\xi_j}\right) v_n(t,\xi) = f(t,\xi), \ \ t_n \le t \le 1,$$

with

$$L\left(t, \xi, D_{\xi_j}\right) = \Psi(t) \sum_{j=1}^{N} D_{\xi_i}^{2m} + \Phi(t) \sum_{j=1}^{n} \xi_j D_{\xi_j}. \tag{2.5}$$

Here

$$\begin{cases} \Psi(t) = \dfrac{1}{\varphi^{2m}(t)} t^{\alpha-1}, \\ \\ \Phi(t) = \dfrac{\varphi'(t)}{\varphi(t)} t^{\alpha-1}, \end{cases} \tag{2.6}$$

and

$$f = t^{\alpha-1} g.$$

Next, we give the following results describing the effect of the inverse change of variables.

Lemma 2.1. *Let $0 < \theta < 1$. Then*

1. $h \in C^{\theta}([0,1]; C(\Omega)) \Rightarrow g \in C^{\theta}([0,1]; C(D))$
2. $g \in C^{\theta}([0,1]; C(D)) \Rightarrow h \in C^{\theta}_{\varphi}([0,1]; C(\Omega))$ *where*

$$C^{\theta}_{\varphi}([0,1]; C(\Omega)) = \left\{ h \in C^{\theta}([0,1]; C(\Omega)) : (\varphi)^{\theta} h \in C^{\theta}([0,1]; C(\Omega)) \right\}.$$

Proof. See Proposition 3.1 in [5]. □

We have also the following results concerning the smoothness of coefficients of our elliptic operator L given by (2.5).

Lemma 2.2. *Let $\Psi(.)$ and $\Phi(.)$ be the real valued functions given by (2.6). Then,*

1. $\Psi(.)$ *and* $\Phi(.)$ *are in* $C^1([t_n, 1])$,
2. $\Psi(.)$ *and* $\Phi(.)$ *are in* $C^{\theta}([t_n, 1])$, $0 < \theta < 1$.

To establish the abstract version of our problem, we consider the following vector-valued functions:

$$\begin{aligned} v_n &: [t_n, 1] \to E;\ t \longrightarrow v_n(t); \quad v_n(t)(\xi) = v_n(t, \xi), \\ f &: [t_n, 1] \to E;\ t \longrightarrow f(t); \quad f(t)(\xi) = f(t, \xi), \end{aligned}$$

where $E = C(\Omega)$ and the abstract version of it is given by

$$v_n'(t) + A(t) v_n(t) = f(t),\ t \geq t_n, \tag{2.7}$$

under

$$v_n(t_n) = 0, v_n(1) = 0 \tag{2.8}$$

and $(A(t))_{t_n \leq t \leq T}$ is a family of closed linear operators with domains $D(A(t))$ (which are not dense) defined by

$$\begin{cases} D(A(t)) := \left\{ w \in W_0^{2m,p}(\Omega) \cap C(\Omega),\ p > 2, m \in \mathbb{N}^* : L\left(t, \xi, D_{\xi_j}\right) w \in C(\Omega) \right\}, \\ (A(t)) w(\xi) := L\left(t, \xi, D_{\xi_j}\right) w(\xi). \end{cases} \tag{2.9}$$

Without loss of generality, consider the following natural change of function

$$w(t) = v_n(t + t_n) \text{ and } g(t) = f(t + t_n);$$

Then, we are concerned with following problem

$$w'(t) + A(t)\, w(t) = g(t)\,,\; t \in [0.1]\,, \tag{2.10}$$

under

$$w(0) = 0, w(1) = 0 \tag{2.11}$$

The spectral properties of the family of operators (2.9) are summarized in the following proposition

Proposition 2.1. *1. $A(t)$ is a family of a closed non densely defined operator satisfying the Krein-ellipticity property, that is:*

$$\rho(A(t)) \supset \Sigma_{\omega,\upsilon_0} = \{\lambda \in \mathbb{C}\backslash\{\omega\} \;/\; \upsilon_0 \leqslant |\arg(\lambda - \omega)| \leqslant 2\pi - \upsilon_0\} \tag{2.12}$$

(here $\rho(A(t))$ is the resolvent set of $A(t)$) and

$$\sup_{\lambda \in \Sigma_{\omega,\upsilon_0}} \left\|(\lambda - \omega)(A(t) - \lambda I)^{-1}\right\|_{L(E)} < +\infty, \tag{2.13}$$

for some given $\omega \in \mathbb{R}$ and $\upsilon_0 \in]0, \pi/2[$.

2. There exists $C > 0$ such that

$$\forall\lambda \geqslant 0, \forall t \geq 0 \quad \left\|(A(t) - \lambda I)^{-1}\right\|_{L(E)} \leqslant \frac{C}{1+\lambda}. \tag{2.14}$$

3. The operator-valued function $t \mapsto (A(t)-\lambda I)^{-1}$ defined on $[0,1]$ is in $C^1([0,1]\,; L(E))$ we suppose that there exists $C > 0$ such that

$$\forall\lambda \geqslant 0, \forall t \geq 0 \quad \left\|\frac{\partial}{\partial t}(A(t) - \lambda I)^{-1}\right\|_{L(E)} \leqslant \frac{C}{1+\lambda}, \tag{2.15}$$

and there exists $K > 0$ such that for all $t > \tau \geq 0$:

$$\left\|\frac{\partial}{\partial t}(A(t) - \lambda I)^{-1} - \frac{\partial}{\partial \tau}(A(\tau) - \lambda I)^{-1}\right\|_{L(E)} \leq \frac{K\,|t-\tau|^{\theta}}{1+\lambda}. \tag{2.16}$$

Proof. See Proposition 7.10 and Proposition 7.11 in [2]. □

Our purpose is to establish some results for Problem (2.7). Recall here that a strict solution is a function w_n such that function v such that for every $t \geq t_n$

$$w \in C^1([0,1]\,, E) \cap C([0,1[\,, D(A(t))),$$

furthermore v satisfies conditions (2.8).

The techniques used here are essentially based on the Dunford functional calculus and the methods applied in [2]. We know that if $A_n(t)$ is a constant operator satisfying (2.14), the representation of the solution w_n is given by the formula

$$w(t) = -\frac{1}{2i\pi}\int_{\Gamma}\int_0^t \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\;(A - \lambda I)^{-1} g(s)ds\; d\lambda$$

$$-\frac{1}{2i\pi}\int_{\Gamma}\int_{t}^{1}\frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ (A-\lambda I)^{-1}g(s)ds\ d\lambda.$$

Here Γ is the boundary of Σ_{ω,v_0} oriented from ∞e^{+iv_0} to ∞e^{-iv_0}. Keeping in mind the constant case, we look for a solution of Problem (2.10)-(2.11) in the following form :

$$w(t)=-\frac{1}{2i\pi}\int_{\Gamma}\int_{0}^{t}\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ (A(t)-\lambda I)^{-1}g^{*}(s)ds\ d\lambda$$
$$-\frac{1}{2i\pi}\int_{\Gamma}\int_{t}^{1}\frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ (A(t)-\lambda I)^{-1}g^{*}(s)ds\ d\lambda,$$

where g^* is an unknown function to be determined in some adequate space in order to obtain a strict solution w. The following results describe some interesting properties of the vector valued function w.

Proposition 2.2. *Suppose that $g^{*}\in C^{\theta}([0,1],E)$, $0<\theta<1$. Then, under Assumptions (2.14)-(2.15)-(2.16), we have for all $t\in[0,1]$:*

$$w(.)\in C([0,1];D(A(.)).$$

Proof. It suffices to show that the integral

$$\begin{aligned}I&=-\frac{1}{2i\pi}\int_{\Gamma}\int_{0}^{t}\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ A(t)(A(t)-\lambda I)^{-1}g^{*}(s)ds\ d\lambda\\&-\frac{1}{2i\pi}\int_{\Gamma}\int_{t}^{1}\frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ A(t)(A(t)-\lambda I)^{-1}g^{*}(s)ds\ d\lambda,\\&=I_1+I_2,\end{aligned}\tag{2.17}$$

converges. The two integrals are treated in the same way. Then, we focus on the first one . We write

$$g^{*}(s)=g^{*}(s)-g^{*}(t)+g^{*}(t),$$

this implies that

$$I_1=\Delta_1+\Delta_2,$$

with

$$\Delta_1=-\frac{1}{2i\pi}\int_{\Gamma}\int_{0}^{t}\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ A(t)(A(t)-\lambda I)^{-1}\left(g^{*}(s)-g^{*}(t)\right)ds\ d\lambda,$$

and

$$\Delta_2=-\frac{1}{2i\pi}\int_{\Gamma}\int_{0}^{t}\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ A(t)(A(t)-\lambda I)^{-1}g^{*}(t)ds\ d\lambda.$$

Then,

$$\|\Delta_1\|_E \le \frac{1}{2\pi}\int_\Gamma \frac{1}{\left|\sinh\frac{\lambda}{2}\right|}\left(\int_0^t e^{-Re\lambda\left(t-s-\frac{1}{2}\right)}\left\|A(t)\left(A(t)-\lambda I\right)^{-1}\right\|_{L(E)}(s-t)^\theta ds\right)|d\lambda|$$

$$\le C\int_\Gamma\left(\int_0^t\left|\frac{e^{-Re\frac{\lambda}{2}}e^{-Re\lambda\left(t-s+\frac{1}{2}\right)}}{(1-e^{Re\lambda})}\right|(s-t)^\theta ds\right)|d\lambda|$$

$$\le C\int_\Gamma\left(\int_0^1\left|\frac{e^{-Re\frac{\lambda}{2}}e^{-Re\lambda\left(t-s+\frac{1}{2}\right)}}{\left(1-e^{-\frac{\pi}{2\tan\left(\frac{\pi}{2}-\frac{\theta_0}{2}\right)}}\right)^2}\right|(s-t)^\theta ds\right)|d\lambda|$$

$$\le \frac{C}{\left(1-e^{-\frac{\pi}{2\tan\left(\frac{\pi}{2}-\frac{\upsilon_0}{2}\right)}}\right)^2}\int_\Gamma\left(\int_0^1 e^{-Re\lambda}e^{-Re\lambda(t-s)}(s-t)^\theta ds\right)|d\lambda|$$

$$\le \frac{C}{Re\lambda}.$$

For the second quantity, one has

$$\|\Delta_2\|_E$$

$$\le \frac{1}{2\pi}\int_\Gamma\int_0^t\left|\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\right|\left\|A(t)\left(A(t)-\lambda I\right)^{-1}\right\|_{L(E)}\|g^*(t)\|_E\, ds\, d\lambda$$

$$\le \frac{C}{\left(1-e^{-\frac{\pi}{2\tan\left(\frac{\pi}{2}-\frac{\theta_0}{2}\right)}}\right)^2}\int_\Gamma\left(\int_0^1 e^{-Re\frac{\lambda}{2}}e^{-Re\lambda\left(t-s+\frac{1}{2}\right)}ds\right)|d\lambda|$$

$$\le \frac{C}{Re\lambda}.$$

□

Proposition 2.3. *Suppose that* $g^*\in C^\theta([0,1],E)$, $\theta\in]0,1[$. *Then, under Assumptions (2.14), we have :*

$$w(.)\in C^1([0,1];E).$$

Proof. Let $0\le\tau<t\le1$. We have

$$w(t)-w(\tau)=\Pi_1+\Pi_2+\Pi_3,$$

where

$$\Pi_1=-\frac{1}{2i\pi}\int_\Gamma\int_\tau^t\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\left(A(t)-\lambda I\right)^{-1}g^*(s)ds\,d\lambda$$

$$+\frac{1}{2i\pi}\int_\Gamma\int_\tau^t\frac{e^{-\lambda\left(\tau-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\left(A(\tau)-\lambda I\right)^{-1}g^*(s)ds\,d\lambda$$

$$\Pi_2 = -\frac{1}{2i\pi}\int_\Gamma \int_t^1 \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$+\frac{1}{2i\pi}\int_\Gamma \int_t^1 \frac{e^{-\lambda\left(\tau-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(\tau)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$

and

$$\Pi_3 = +\frac{1}{2i\pi}\int_\Gamma \int_0^\tau \frac{e^{-\lambda\left(\tau-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(\tau)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$-\frac{1}{2i\pi}\int_\Gamma \int_0^\tau \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda,$$

The proof is very technical and the calculus are very cumbersome, we just treat the quantity Π_1. We write

$$\Pi_1 = -\frac{1}{2i\pi}\int_\Gamma \int_\tau^t \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$+\frac{1}{2i\pi}\int_\Gamma \int_\tau^t \frac{e^{-\lambda\left(\tau-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$-\frac{1}{2i\pi}\int_\Gamma \int_\tau^t \frac{e^{-\lambda\left(\tau-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$+\frac{1}{2i\pi}\int_\Gamma \int_\tau^t \frac{e^{-\lambda\left(\tau-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(\tau)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda,$$

then,

$$\Pi_1 = -\frac{1}{2i\pi}\int_\Gamma \int_\tau^t \left(\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} - \frac{e^{-\lambda\left(\tau-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\right) \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$-\frac{1}{2i\pi}\int_\Gamma \int_\tau^t \frac{e^{-\lambda\left(\tau-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(\left(A(t)-\lambda I\right)^{-1} - \left(A(\tau)-\lambda I\right)^{-1}\right) g^*(s)ds\ d\lambda,$$

then

$$\Pi_1 = -\frac{1}{2i\pi}\int_\Gamma \frac{1}{\sinh\frac{\lambda}{2}} \left(\int_\tau^t \left(e^{-\lambda\left(t-s-\frac{1}{2}\right)} - e^{-\lambda\left(\tau-s-\frac{1}{2}\right)}\right) \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\right) d\lambda$$
$$-\frac{1}{2i\pi}\int_\Gamma \frac{1}{\sinh\frac{\lambda}{2}} \left(\int_\tau^t e^{-\lambda\left(\tau-s-\frac{1}{2}\right)} \left(\left(A(t)-\lambda I\right)^{-1} - \left(A(\tau)-\lambda I\right)^{-1}\right) g^*(s)d\right) s\ d\lambda$$

$= \Pi_{11} + \Pi_{12}.$

Taking into account the estimate (2.14), a direct computation show that

$$\|\Pi_{11}\|_E \leq C\,|t-\tau|$$

For Π_{12}, since the operator-valued function $t \mapsto (A(t)-\lambda I)^{-1}$ *defined on* $[0,1[$ *is in* $C^1([0,1]\,;L(E))$, it follows then that

$$\|\Pi_{12}\|_E \leq C\,|t-\tau|\,.$$

□

Following the same reasoning and techniques, we obtain

Proposition 2.4. *Suppose that* $g^* \in C^\theta([0,1]\,,E)$, $\theta \in]0,1[$. *Then, under Assumptions (2.14)-(2.15)-(2.16), we have for all* $t \geq 0$:

$$w'(.) \in C^1([0,1]\,;E).$$

Proposition 2.5. *Suppose that* $g^* \in C^\theta([0,1]\,,E)$, $\theta \in]0,1[$. *Then, under Assumptions (2.14)-(2.15)-(2.16), the function* w *defined by (2.17) satisfying the following equation given by*

$$w'(t) = -A(t)\,w(t) + g^*(t) - R_\lambda(g^*)(t),$$

where

$$R_\lambda(g^*)(t) = \frac{1}{2i\pi}\int_\Gamma \int_0^t \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ \frac{\partial}{\partial t}\left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$+\frac{1}{2i\pi}\int_\Gamma \int_t^1 \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ \frac{\partial}{\partial t}\left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda.$$

Proof. Using the same argument as in [5], set

$$w_\varepsilon(t) = -\frac{1}{2i\pi}\int_\Gamma \int_0^{t-\varepsilon} \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$-\frac{1}{2i\pi}\int_\Gamma \int_{t+\varepsilon}^1 \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$

Then, a direct computation show that

$$w'_\varepsilon(t) = \frac{1}{2i\pi}\int_\Gamma \int_0^{t-\varepsilon} \lambda\frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ \left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$
$$-\frac{1}{2i\pi}\int_\Gamma \int_0^{t-\varepsilon} \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\ \frac{\partial}{\partial t}\left(A(t)-\lambda I\right)^{-1} g^*(s)ds\ d\lambda$$

$$- \frac{1}{2i\pi} \int_{\Gamma} \frac{e^{-\lambda\left(t-t+\varepsilon-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^{*}(t-\varepsilon) d\lambda$$
$$+ \frac{1}{2i\pi} \int_{\Gamma} \int_{t+\varepsilon}^{1} \lambda \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^{*}(s) ds\, d\lambda$$
$$- \frac{1}{2i\pi} \int_{\Gamma} \int_{t+\varepsilon}^{1} \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \frac{\partial}{\partial t} \left(A(t)-\lambda I\right)^{-1} g^{*}(s) ds\, d\lambda$$
$$+ \frac{1}{2i\pi} \int_{\Gamma} \frac{e^{-\lambda\left(t-t-\varepsilon+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t)-\lambda I\right)^{-1} g^{*}(t-\varepsilon) d\lambda$$

Now, the use of the following algebraic identity

$$\left(A(t)-\lambda I\right)^{-1} = \frac{A(t)\left(A(t)-\lambda I\right)^{-1} - I}{\lambda},$$

allows us to write $w'_{\varepsilon}(t)$ as follows

$$w'_{\varepsilon}(t) = Q_{\varepsilon,1}(t) + Q_{\varepsilon,2}(t) + Q_{\varepsilon,3}(t) + Q_{\varepsilon,4}(t),$$

where

$$Q_{\varepsilon,1}(t) = \frac{1}{2i\pi} \int_{\Gamma} \int_{0}^{t-\varepsilon} \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} A(t)\left(A(t)-\lambda I\right)^{-1} g^{*}(s) ds\, d\lambda$$
$$+ \frac{1}{2i\pi} \int_{\Gamma} \int_{t+\varepsilon}^{1} \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} A(t)\left(A(t)-\lambda I\right)^{-1} g^{*}(s) ds\, d\lambda$$

$$Q_{\varepsilon,2}(t) = +\frac{1}{2i\pi} \int_{\Gamma} \frac{e^{-\lambda\left(\frac{1}{2}-\varepsilon\right)}}{\lambda \sinh\frac{\lambda}{2}} A(t)\left(A(t)-\lambda I\right)^{-1} g^{*}(t-\varepsilon) d\lambda$$
$$- \frac{1}{2i\pi} \int_{\Gamma} \frac{e^{-\lambda\left(\varepsilon-\frac{1}{2}\right)}}{\lambda \sinh\frac{\lambda}{2}} A(t)\left(A(t)-\lambda I\right)^{-1} g^{*}(t-\varepsilon) d\lambda$$

$$Q_{\varepsilon,3}(t) = -\frac{1}{2i\pi} \int_{\Gamma} \int_{0}^{t-\varepsilon} \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \frac{\partial}{\partial t} \left(A(t)-\lambda I\right)^{-1} g^{*}(s) ds\, d\lambda$$
$$- \frac{1}{2i\pi} \int_{\Gamma} \int_{t+\varepsilon}^{1} \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \frac{\partial}{\partial t} \left(A(t)-\lambda I\right)^{-1} g^{*}(s) ds\, d\lambda$$

and

$$Q_{\varepsilon,4}(t) = \frac{1}{2i\pi} \int_{\Gamma} \frac{e^{-\lambda\left(\varepsilon-\frac{1}{2}\right)}}{\lambda \sinh\frac{\lambda}{2}} g^{*}(t-\varepsilon) d\lambda - \frac{1}{2i\pi} \int_{\Gamma} \frac{e^{-\lambda\left(\frac{1}{2}-\varepsilon\right)}}{\lambda \sinh\frac{\lambda}{2}} g^{*}(t-\varepsilon) d\lambda$$

Lebesgue's and Cauchy's theorems give us

$$\lim_{\varepsilon\to 0} Q_{\varepsilon,1}(t) = -\frac{1}{2i\pi}\int_\Gamma A(t)\left(\int_0^t \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} + \int_t^1 \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}}\right)(A(t)-\lambda I)^{-1} g^*(s)ds\, d\lambda$$
$$= -A(t)\, w(t),$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} Q_{\varepsilon,2}(t) &= +\frac{1}{2i\pi}\int_\Gamma \frac{e^{-\frac{\lambda}{2}}-}{\lambda\sinh\frac{\lambda}{2}} A(t)(A(t)-\lambda I)^{-1} g^*(t)d\lambda \\ &- \frac{1}{2i\pi}\int_\Gamma \frac{e^{+\frac{\lambda}{2}}}{\lambda\sinh\frac{\lambda}{2}} A(t)(A(t)-\lambda I)^{-1} g^*(t)d\lambda \\ &= A(t)(A(t))^{-1} g^*(t) \\ &= g^*(t).\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} Q_{\varepsilon,3}(t) &= -\frac{1}{2i\pi}\int_\Gamma\int_0^t \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \frac{\partial}{\partial t}(A(t)-\lambda I)^{-1} g^*(s)ds\, d\lambda \\ &- \frac{1}{2i\pi}\int_\Gamma\int_t^1 \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \frac{\partial}{\partial t}(A(t)-\lambda I)^{-1} g^*(s)ds\, d\lambda\end{aligned}$$

and

$$\lim_{\varepsilon\to 0} Q_{\varepsilon,4}(t) = \frac{1}{2i\pi}\int_\Gamma \frac{e^{-\lambda\left(\varepsilon-\frac{1}{2}\right)}}{\lambda\sinh\frac{\lambda}{2}} g^*(t-\varepsilon)d\lambda - \frac{1}{2i\pi}\int_\Gamma \frac{e^{-\lambda\left(\frac{1}{2}-\varepsilon\right)}}{\lambda\sinh\frac{\lambda}{2}} g^*(t-\varepsilon)d\lambda = 0.$$

Summing up, we conclude that

$$w'(t) = \lim_{\varepsilon\to 0} w'_\varepsilon(t) = -A(t)w(t) + g^*(t) + R_\lambda(g^*)(t),$$

where

$$\begin{aligned}R_\lambda(g^*)(t) &= -\frac{1}{2i\pi}\int_\Gamma\int_0^t \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \frac{\partial}{\partial t}(A(t)-\lambda I)^{-1} g^*(s)ds\, d\lambda \\ &- \frac{1}{2i\pi}\int_\Gamma\int_t^1 \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \frac{\partial}{\partial t}(A(t)-\lambda I)^{-1} g^*(s)ds\, d\lambda.\end{aligned}$$

□

The natural question that arises, what is the relation between the vectorial functions g^* and g, the answer is given by the following theorem

Proposition 2.6. *Suppose that $g \in L^\infty([0,1];E)$. Then, there exists $\omega^* \in \mathbb{R}^*$ such that for all $\lambda \in \rho(A(.))$ with $|\lambda| > |\omega^*|$, the equation*

$$g(.) = g^*(.) - R_\lambda(g^*)(.),$$

admits a unique solution $g^* \in L^\infty([0,1];E)$.

Proof. The result is handled by proving that

$$g^*(.) = (1 - R_\lambda)^{-1} g(.),$$

To do this, it suffices to show that

$$\|R_\lambda\| < 1.$$

Using Assumption (2.15), it is easy to deduce that

$$\|R_\lambda\|_{L(L^\infty([0,1];E))} = O\left(\frac{1}{Re\lambda}\right),$$

Now, choosing a suitable sufficiently large ω^*, we conclude that for any $|\lambda| > |\omega^*|$

$$\|R_\lambda\|_{L(L^\infty([0,1];E))} < 1.$$

□

As a direct consequence of Assumption (2.16), we obtain the following result describing the regularity of the operator R_λ

Proposition 2.7. *Suppose that $g^* \in C^\theta([0,1], E)$, $\theta \in]0,1[$. Then, under Assumptions (2.16), one has*

$$R_\lambda(g^*)(.) \in C^\theta([0,1], E).$$

Keeping in mind the key estimates (2.14)-(2.15) and (2.16), we are able to justify our main results for our translated abstract Problem (2.10)-(2.11)

Theorem 2.1. *Suppose that $g^* \in C^\theta([0,1], E)$, $\theta \in]0,1[$. Then, there exists $\omega^* \in \mathbb{R}^*$ such that for all $\lambda \in \rho(A(t))$ with $|\lambda| > |\omega^*|$, the the function*

$$w(t) = -\frac{1}{2i\pi}\int_\Gamma \int_0^t \frac{e^{-\lambda\left(t-s-\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t) - \lambda I\right)^{-1} g^*(s)ds\ d\lambda$$

$$-\frac{1}{2i\pi}\int_\Gamma \int_t^1 \frac{e^{-\lambda\left(t-s+\frac{1}{2}\right)}}{\sinh\frac{\lambda}{2}} \left(A(t) - \lambda I\right)^{-1} g^*(s)ds\ d\lambda$$

is the unique strict solution of Problem Problem (2.10)-(2.11) satisfying

$$w(t),\ A(t)w(t) \in C^\theta([0,1], E).$$

Remark 2.1. Using the estimates (2.16), We easily deduce the existence of $C > 0$ such that

$$\max_{t\in[0,1]} |w(t) \leq C|. \tag{2.18}$$

At this level, we can conclude that our Problem (2.7)-(2.8) has a unique strict solution of is given by

$$v_n(t) = w(t - t_n).$$

Furthermore, thanks to (2.18), we can extract a convergent subsequence

$$v_{n_j}(t) = w\left(t - t_{n_j}\right)$$

where

$$\left(t_{n_j}\right) \to 0.$$

Then, after a passage to the limit, we deduce the following important result

Theorem 2.2. *Suppose that* $f \in C^{\theta}([0,1], E)$*,* $\theta \in]0,1[$*. Then, Problem Problem*

$$v_n'(t) + A(t) v_n(t) = f(t), \, t \in [0,1], \tag{2.19}$$

$$v_n(0) = 0, v_n(1) = 0 \tag{2.20}$$

has a unique strict solution satisfying

$$v(t), \; A(t)v(t) \in C^{\theta}([0,1], E).$$

Since Problem (2.19)-(2.20) can be viewed as the abstract version of our original problem (1.1)-(1.3)-(1.4). Then, our main result for this problem is as follows

Theorem 2.3. *Let* $h \in C^{\theta}([0,1]; C(\Omega))$*,* $0 < \theta < 1$*. Then, Problem (1.1)-(1.3)-(1.4) has a unique strict solution* $u \in C^2(\Pi)$*. Moreover,* u *satisfies the following maximal regularity propriety*

$$\begin{cases} \mathcal{D}_t^{\alpha} u(t,x) \in C_{\varphi}^{\theta}([0,1]; C(\Omega)), \\ \\ and \\ \\ \sum_{j=1}^{N} D_{x_i}^{2m} u(t,x) \in C_{\varphi}^{\theta}([0,1]; C(\Omega)). \end{cases}$$

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